An-Najah National University

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Confidence – based Optimization for the Single Period Inventory Control Model

By

Thana'a Hussam eddin Amin Abu Sa'a

Supervisor

Dr. Mohammad Ass'ad

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This thesis was successfully defended on 7/2/2016 and approved by:

Defense Committee Members	**Signature**
– Dr. Mohammad Ass'ad /Supervisor	……………
– Dr. Sa'ed Mallak /External Examiner	……………
– Dr. Ali Barakat /Internal Examiner	……………

Dedication

To Mom and Dad

Acknowledgment

First and foremost I am grateful to Allah (swt) for giving me the strength to complete this thesis.

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Thanks also to all who I see in their eyes how much I am beautiful.

الإقـرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Confidence – based Optimization for the Single Period Inventory Control Model

أقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة اليه حيثما ورد، وأن هذه الرسالة ككل أو جزء منها لم يقدم من قبل لنيل أي درجة أو بحث علمي أو بحثي لدى مؤسسة تعليمية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name: **اسم الطالبة:** فداء حسام الدين أحسن أبو صالح

Signature: **التوقيع:** فداء

Date: **التاريخ:** ٢٠١٦٢/٧

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Abstract

In this thesis we introduce the issue of demand estimation. We study a problem of controlling the inventory of a single item over a single period with stochastic demand in which the distribution of the demand has an unknown parameter.

We assume that the decision maker has a past demand sample and the demand distribution is known but some of its parameters are not known.

We introduce some approaches to estimate the unknown parameter and depending on results from estimating the unknown parameter we identify a range of order quantities that-with $1-\alpha$ confidence coefficient – contains the optimal order quantity, and then we construct an interval for the estimated expected cost that the manager will pay if he orders any quantity from the range of candidate quantities.

We consider three cases, the demand has a Binomial distribution with unknown parameter p , and the demand has a Poisson distribution with unknown parameter λ, also we consider the case in which the demand has an Exponential distribution with unknown parameter λ.

We present numerical examples in order to clarify our strategy and to show how the confidence interval approach complements with the point estimation approach in order to give the best outlook to the manager to take a decision that achieve an optimal profit.

Chapter One

Introduction

'Operations Research' was developed during the World War II, but the scientific origin of the subject dates much further back.

Many definitions of Operations Research are available. The following are a few of them. In the words of T.L Saaty, "operations research is the art of giving bad answers to problem which otherwise have worse answers". According to Fabrycky and Torgersen, "operations research is the application of scientific methods to problems arising from the operations involving integrated system by man, machine and materials. It normally utilizes the knowledge and skill of an interdisciplinary research team to provide the managers of such systems with optimum operating solutions". Churchman, Ackoff and Arnoff observe, "operations research in the most general sense can be characterized as the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the operations with optimum solutions to the problems"[10].In simple words, operations research is the discipline of applying advanced analytical methods to help make better decisions.

Operations research comprises of various branches which include Inventory control, Queuing theory, Mathematical Programming, Game theory and Reliability methods. In all these branches many real life problems are conceptualized as mathematical and stochastic models.

Operations research provides tools to (i) analyze the activity (ii) assist in decision making, (iii) enhancement of organizations and experiences all around us. Application of operations research involves better scheduling of airline crews, the design of waiting lines at Disney theme parks, and global resource planning decisions to optimizing hundreds of local delivery routes. All benefit directly from operations research decision.

Inventory control is one of the most developed fields of operations research. Many sophisticated methods of practical utility were developed in inventory management by using tools of mathematics, stochastic process and probability theory. [10]

The study on inventory control deals with two types of problems such as single-item and multi-item problems. Concerning the process of demand for single-items, the mathematical inventory models are divided into two large categories deterministic and stochastic models

The simplest periodic model is the single period model. The decision problem reduces to only one period. Such inventory problems occur if the products cannot be sold after the period.

Examples of these are fashion articles, travel offers, ticket sales for large presentations and daily newspapers.

Consider a problem of controlling the inventory of a single item over a single period with stochastic demand, this problem is also known as a Newsvendor problem, or the newsboy problem, we need in this problem to find the order quantity which maximizes the expected profit in a single period probabilistic demand.

Early in the morning, the newsboy buys a stack of newspapers and tries to sell these during the course of the day. He can only return the unsold papers at a loss. If he carries only a small quantity of newspapers, he misses the profit. Demand is uncertain but its distribution is known. His decision problem: "How many newspapers do I buy to maximize my profit expectations?" [2]

And so the manager faces costs if he orders too much or if he orders too little. This problem therefore consists of deciding the size of a single order that must be placed before observing demand when there are overage and underage costs. And so the objective is to decide the optimal order quantity Q so that the expected total cost is minimized.

Most of the research on single-period inventory models has focused on the case in which demand distribution parameters are known, but in this thesis we will consider the situation in which the parameter of such distribution is not known, it is clear that the applicability of these models directly depends on the reliability of demand parameters estimation.

And so we will consider the situation in which the decision maker knows the type of the random demand distribution, but he does not know the actual values of some of the parameter of such a distribution. The decision maker is given a set of M past realizations of the demand. From these realizations he has to infer the optimal order quantity and, he has to estimate the cost associated with the optimal Q^* he has selected.

We will consider two approaches to estimate the parameter of the demand distribution, the first approach is the point estimation approach and the second approach is the confidence interval approach.

In the first approach we will use the maximum likelihood estimator and the Bayes estimator.

In the Bayes estimator we will consider a “prior” distribution, which quantifies the uncertainty in the values of the unknown parameters before the data are observed [29], then update prior distribution with the data using Bayes' theorem to obtain a posterior distribution. The posterior distribution of the parameter is then used to construct, first, the posterior distribution of the demand, and then to derive the optimal order quantity [36] and the objective function, expected cost.

On the other hand, in the maximum likelihood estimator a parametric demand distribution is empirically selected and point estimates for the unknown parameters are obtained according to the observed data [33].

So in our work we will introduce a strategy to address the issue of demand estimation in single-period inventory optimization problem. Consider a possibly very limited set of past demand observations. The strategy would analyze these data and provide a single most-promising order quantity and an estimated cost associated with it. Unfortunately, both the maximum likelihood estimator and the Bayes estimator ignore the uncertainty around the estimated order quantity and its associated expected total cost or profit.

In the second approach, we will try to clarify an approach that employs exact confidence interval in order to identify a range of candidate order quantities that includes the real optimal order quantity for the underlying stochastic demand process with unknown parameters, with a certain confidence probability. In addition, for each candidate order quantity that is identified, this approach computes upper and lower bounds for the associated cost. This range covers the actual cost, the decision maker will face if he selects that particular quantity. The approach we will consider does not simply provide point estimation; it provides instead complete information to the decision maker about the set of potentially optimal order quantities according to the available data and to the chosen confidence level and about the confidence interval for the expected cost associated with each of these quantities.

In the situation where the demand has, for example, a binomial distribution we will consider the parameter p, a success probability in the binomial distribution, is unknown .The decision maker is given a set of M past of the realizations of the demand, we try to establish exact confidence interval for the binomial distribution. This method uses the binomial cumulative distribution function in order to build the interval from the data observed.

We will try to compute upper and lower bounds for the optimal order quantity in our problem under partial information. First, we will construct the confidence interval for the unknown parameter p of the binomial demand, and then depending on the confidence interval for the unknown parameter p, we will try to consider a set that the optimal order quantity is a member of it. After that we will try to compute upper and lower bounds for the cost that a manager will pay, with confidence probability.

Another case, we will consider the situation where the demand has a Poisson distribution, in which the parameter λ, rate of Poisson demand, is unknown. As the previous case, the decision maker is given a set of M past of the realizations of the demand; we will estimate λ using the confidence interval that was proposed by Garwood [36]. We will take the similar fashion as in the binomial case for computing a set that contains the optimal quantity and the interval for the associated cost.

Finally, Numerical examples are presented in which the researcher shows how the two approaches are complements with each other. Our aim is to establish a confidence ratio that the decision from discussed approaches is not worse.

The strategy of our investigation in this thesis is as follows:

We start from the basic concepts of single period inventory control model and some basic concepts from probability theory.

In chapter three, we will clarify how we can estimate the unknown parameter using the point estimation and the confidence interval estimation. Then in chapter four, we will analytically combine parameter estimation analysis and inventory optimization. Finally, we will give numerical examples and the summary of our main results and conclusions.

Objective:

In our research we will employ confidence interval approach to find a range of candidate order quantities that include the actual optimal order quantity for a single item with stochastic demand over a single period with unknown parameter and with certain confidence probability. Then apply this approach to three demand distribution: binomial, Poisson, and exponential.

Methodology:

Clarifying how we can combine confidence interval analysis and inventory optimization. Implementing approaches for each distribution demand in order to compute intervals that involve the optimal order quantity- with confidence probability. Then we will present numerical examples in order to show how this approach complements with other approaches.

Chapter Two

Inventory Control Model

2.1General Inventory Control Model

Inventory one of the most expensive assets of many companies, representing as much as 50% of total invested capital [32].

Inventory is a quantity or store of goods that is held for some purpose. Also it is the stock of any item or resource used in an organization and can include: raw materials, finished products, and component parts. In other words, inventory is the stock of resources that is used to satisfy the current or the future needs. Inventory control, is an attempt to balance inventory needs and requirements with the need to minimize costs resulting from obtaining and holding inventory [3].

An inventory system is the set of policies and controls that monitor the answers of the inventory decision questions “when and how much to order?”

Inventory control systems aim to ensure that you have a sufficient supply of whatever the manager sells to meet expected demand, while at the same time avoiding ordering mistakes, resulting in costly understock and overstock situations. Inventory control faces special challenges for companies that operate on a "single-period" inventory model, in which the manager get only one chance to order in the stand at a time period.

Periods and inventories:

To illustrate the idea of the different inventories, say the manager who own a coat store, and he has 20 coats of brown color in stock. If he doesn't sell them today, he can sell them tomorrow or the next day. Even if models are changing, he can probably discount the coats enough to get them sold. This is the typical inventory model. Now imagine a newspaper vendor. The newsboy orders a certain number of newspapers from the publisher, the publisher brings them in the morning , and the newsboy sell them during the day. But exceeds of them can't be rolled over to the next day. At day's end, those papers have no value. This is a single-period inventory model [6].

Similarly, other items such as fashions are sold at a loss simply because there is no storage space or it is uneconomical to keep them for the next year [9].

Some purposes of Inventory:

1. To maintain independence of operations
2. To meet variation in product demand
3. To allow flexibility in production scheduling
4. To provide a protection for variation in raw material delivery time
5. To take advantage of economic purchase order size

Inventory control serves several important functions and adds a great deal of flexibility to the operation of a firm. As discussed in [32] there are five main uses of inventory:

1. The decoupling function: Inventory can act as a buffer to avoid the delays and inefficiencies.

2. Storing resources: Resources can be stored as work-in-process or as finished product.
3. Irregular supply and demand: Inventory helps when there is irregular supply or demand.
4. Quantity discounts: lower unit cost due some times to large purchased (produced quantities).
5. Avoiding partially stock outs and shortages: If a company is repeatedly or some times out of stock, customers are likely to go elsewhere to satisfy their needs. Lost goods can be an expensive price to pay for not having the right item at the right time.

The manager uses operations research to improve their inventory policy by using scientific inventory management comprising the following steps:

1. Formulate a mathematical model describing the behavior of the inventory system.
2. Seek mathematically an optimal inventory policy with respect to this model.
3. Use a computerized information processing system to maintain a record of the current inventory levels.
4. Using this record of current inventory levels, apply the optimal inventory policy to signal when and how much to replenish inventory [17].

Types of Inventory Systems Models (by the degree of certainty of data)

- Deterministic model: has a complete certainty and all information needed are available with fixed and known values. Example:

Economic Order Quantity (EOQ), which the parameter demand is known.

- Probabilistic (stochastic) Inventory model: the parameter (expected) demand is known and some of data is not known with certainty and take into account that information will be available after the decision is made. Examples: single-period order quantity, reorder-point quantity and periodic-review order quantity.

The basis for solving inventory models is the minimization of the following inventory expected cost function:

Total inventory expected cost = Purchasing cost+ setup cost+ expected holding cost+ expected shortage cost.

Such that:

1. Purchasing cost is the price per unit of an inventory item. At times the item is offered at a discount if the order size exceeds a certain amount, which is a factor in deciding how much to order.
2. Setup cost represents the fixed charge incurred when an order is placed regardless of its size. Increasing the order quantity reduces the setup cost associated with a given demand, but will increase the average inventory level and hence the cost of tied capital. On the other hand, reducing the order size increases the frequency of ordering and the associated setup cost. An inventory cost model balances the two costs.

3. Holding cost represents the cost of maintaining inventory in stock. It includes the interest on capital and the cost of storage, maintenance, and handling.
4. Shortage cost is the penalty incurred when we run out of stock. It includes potential loss of income and the more subjective cost of loss in customer's goodwill. When a customer seeks the product and finds the inventory empty, the demand can either go unfulfilled or be satisfied later when the product becomes available. The former case is called a lost sale, and the latter is called a backorder.

An inventory system may be based on periodic review (e.g., ordering every week or every month), in which new orders are placed at the start of each period. Alternatively, the system may be based on continuous review, where a new order is placed when the inventory level drops to a certain level, called the reorder point. The EOQ is used as part of a continuous review inventory system in which the level of inventory is monitored at all times and a fixed quantity is ordered each time the inventory level reaches a specific reorder point.

The EOQ provides a model for calculating the appropriate reorder point and the optimal reorder quantity to ensure the instantaneous replenishment of inventory with no shortages. It can be a valuable tool for small business owners who need to make decisions about how much inventory to keep on hand, how many items to order each time, and how often to reorder to incur the lowest possible costs. [22]

An example of periodic review can occur in a gas station where new deliveries arrive at the start of each week.

Continuous review occurs in retail stores where items (such as cosmetics) are replenished only when their level on the shelf drops to a certain level [40].

2.2 Basic Concepts from Probability Theory

This section is considered to clarify some basic concepts from probability theory and discussed a number of important distributions which have been found useful for our work.

- Random Variable: A random variable, usually written as X, is a variable, whose value is subject to variations due to chance [38] and its possible values are numerical outcome of a random phenomenon. There are two types of random variables, discrete and continuous. The expected value or mean of X is denoted by $E(X)$ and its variance by $\sigma^2(X)$ where $\sigma(X)$ is the standard deviation of X [18].
- Discrete random variable: A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, 4,… Examples for discrete random variables include the number of children in a family, the number of patients in a doctor's clinic and the number of defective light bulbs in a box of ten.
- Continuous random variable: A continuous random variable is one, which takes not countable number of possible values. Continuous random variables are usually measurements. Examples include height, weight, the amount of sugar in an orange and the time required to run

a mile. A continuous random variable is not defined at specific values. Instead, it is defined over an interval of values, and its probability represented by the area under a curve. The probability of observing any single value is equal to zero [10].

Some Probability Distributions

we will discuss a number of important distributions which have been found useful for our study .

1. Poisson Distribution

The probability distribution of a Poisson random variable X with parameter λ which is representing the average number of successes occurring in a given time interval or a specified region of space is given by the formula [18]:

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \; k = 0,1,2,...$$

For the Poisson distribution we have:

$$E(X) = \sigma^2 (X) = \lambda$$

2. Binomial Distribution

The binomial distribution is a discrete distribution described by the following relationship [39]:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0,1,2,...,n$$

Where p is the probability of success on each trail.

For the Binomial distribution we have:

$$E(X) = np\ ,\ \sigma^2(X) = np(1-p)$$

3. Exponential Distribution

We usually say that the random variable has an Exponential distribution if its probability density function is defined by [39]:

$$P(X = k) = \begin{cases} \lambda e^{-\lambda k} & 0 \le k < \infty \\ 0 & k < 0 \end{cases}$$

Where the parameter $\lambda > 0$

For the Exponential distribution we have:

$$E(X) = \frac{1}{\lambda}\ ,\ \sigma^2(X) = \frac{1}{\lambda^2}$$

2.3 Single period inventory control model

Single item inventory models occur when an item is ordered only once to satisfy the demand for a specified period of time [40].

Consider a single-period order quantity model (sometimes called the newsboy problem or inventory system of perishable goods) this model deals with items of short life and the demand is probabilistic.

Single-period order quantity model means that inventory is not carried over to another period. Furthermore, any remaining products at the end of the period can be disposed of at a certain expense, or can be sold at a lower price than the market price. Initially, this type of modeling was applied to products with very high perishability, such as newspapers. Later, especially in the fashion industry, newsboy models were proven to be of use (Fisher and Raman (1996) who study the single period setting in the fashion industry), and following the decrease of product life cycles in high-tech, such as personal computers and mobile phones. Newsboy models are now well-accepted to model ordering decisions in these environments [27].

This model also has wide applicability in service industries such as airlines and hotels where the key decision is capacity which cannot be stored and the product is generally perishable [11].

The problem of the classical single-period, single-item is to decide on the ordering quantity before market demand is known, so that at the time of ordering demand is uncertain. The objective is to maximize expected profit. If demand D were known at the time of ordering, it is easy to see the optimal decision for the newsvendor. However, since demand is not known at the time of ordering, the problem becomes more difficult. The demand D has to be understood as a random variable with a known demand distribution. In fact, since for real problems the exact demand distribution cannot be known either, it has to be well estimated based on collected random observations from the past. Demand can then be described by its corresponding cumulative distribution function (cdf) $F(x) = p(D \le x)$ and probability density function (pdf) $f(x)$ .Since demand cannot be negative, clearly $F(x)$ = 0 for any x < 0 [11].

The classical single-period problem researchers have followed two approaches to solving the SPP. In the first approach, the expected costs of overestimating and underestimating demand are minimized. In the second approach, the expected profit is maximized. Both approaches yield the same results [13]. We use the first approach in stating the single-period problem.

Now, we take a newsvendor as an example to explain the single period inventory model.

The owner of the newspaper stand needs to order newspapers at the beginning of one day, and he has to make appropriate decision about his inventory level. Since if he buys too many papers, some papers will not be sold and have no value at the end of that day. In contrast, if he buys too few papers he has lost the opportunity of making a higher profit [25].

And so, the decision maker has to make decisions about inventory level over limited period to reduce both lost sales and excess inventory and then to optimize the expected profit.

Notice that, period could be one day, one month or any limited period [21].

Assumptions of our model:

- Demand occur instantaneously at the start of the period immediately after the order is received.
- No setup cost is incurred [40].
- Only one order in time period
- Probabilistic distribution of demand (continuous or discrete).
- Instantaneous replenishment.

Now we will clarify the mathematical structure and the symbols used in the development of the model:

D: random variable representing demand during the period.

Q: order quantity purchased at the beginning of the period.

C: unit cost.

P_r: unit price.

S: the salvage value of each unit left over item.

h: unit overage cost : the cost of buying one unit more than the demand, h=C-S.

g :unit underage (shortage) cost: the cost of buying one unit less than the demand, $g = P_r - C$

As discussed in [39], in order to find the optimal order quantity, assume that the demand is a random variable with probability function $f(D)$ and cumulative distribution function $F(D)$.

Let $G(Q, D)$ be the cost function, the cost which the owner will pay when the demand is D and the Q-units are ordered at the start of the period.

$$G(Q,D) = \begin{cases} h(Q-D), & if\, D < Q \\ g(D-Q), & if\, D \geq Q \end{cases} \qquad (2.1)$$

Such that:

$Q-D$: is a random variable has the same distribution as D , which is equal to the excess demand over the supply at the end of the period.

$D-Q$: is a random variable has the same distribution as D, which is equal to the unsatisfied demand remaining at the end of the period[40].

In the presence of uncertainty, the objective is to minimize the expected cost or to maximize the expected profit.

We will determine the expected value of $G(Q, D)$ with respect to the probability function of the demand and then find the optimal value of Q that minimize the expected cost function $E\big(G(Q,D)\big)$.

Since the demand is a random variable then we need to separate the single period inventory problem into a continuous and a discrete random demand.

Assume we know the demand density function $f(D)$ and thus the cumulative distribution function $F(D)$.

We will present the optimal solution under continuous and discrete demand using the standard cost expression in the next two subsections.

2.3.1 The demand is a continuous random variable

Assume the demand is a continuous random variable with probability density function $f(D)$. As in [30] the expected total cost function is given by:

$$\begin{aligned} E(G(Q,D)) &= \int_0^\infty G(Q,D)\times f(D)\,dD \\ &= h\int_0^Q (Q-D)\times f(D)\,dD + g\int_Q^\infty (D-Q)\times f(D)\,dD \end{aligned} \quad (2.2)\ \ [16]$$

Since this function is convex in Q, then we have a unique minimum for the expected cost. So to find the optimal Q, we use the fundamental theory of calculus, i.e. take the derivatives of the expected cost function with respect to Q and equate it to zero. We find that a necessary condition for a relative maximum or relative minimum at $Q*$ is:

$$F(Q*) = \frac{g}{h+g} \qquad (2.3)$$

Since $\frac{\partial^2 E\left(G\left(Q,D\right)\right)}{\partial Q^2} = (h+g)\,f\left(Q*\right) \geq 0$, we have a minimum at $Q*$. [30]

Since $F(Q) = p(D \leq Q) = \int_0^Q f(D)\,dD$, then we can find $Q*$ by:

$$\int_0^{Q^*} f(D)\,dD = \frac{g}{h+g} \qquad (2.4)$$

The value $R = \frac{g}{g+h}$ is called the "critical ratio" or "critical fractile" and is always between zero and one [16].

2.3.2 The demand is a discrete random variable

When a demand is a discrete random variable in which the probability mass function $f(D)$ is defined only at discrete points, then the associated expected total cost function is:

$$\begin{aligned} E(G(Q,D)) &= \sum_{D=0}^{\infty} G(Q,D) \times f(D) \\ &= h\sum_{D=0}^{Q}(Q-D)\,f(D) + g\sum_{D=Q+1}^{\infty}(D-Q)\,f(D) \end{aligned} \qquad (2.5)$$

This function is convex in Q [40], then we determine the optimal quantity by seeking Q such that the expected total cost function is flat at Q^*.

As discussed in [16] we can find Q^* such that $E(G(Q^*,D))$ is approximately equal to $E(G(Q^*+1,D))$, therefore Q^* is the smallest value of Q's such that:

$$F(Q^*) \geq \frac{g}{g+h}$$

Since $F(Q^*) = \sum_{D=0}^{Q^*} f(D)$ then

$$\sum_{D=0}^{Q^*} f(D) \geq \frac{g}{g+h} \qquad (2.6)$$

Again the value $R=\dfrac{g}{g+h}$ is called the "critical ratio" or "critical fractile" and is always between zero and one.

If we can write $\sum_{D=0}^{Q*} f(D)$ in closed form we can find an analytic formula for $Q*$ if not we can find the optimal quantity $Q*$ with simple search procedure starting at $Q=1$ and increase Q until the relation (2.6) is satisfied.

Also if we cannot write $\sum_{D=0}^{Q*} f(D)$ in closed form , the researcher try to find $Q*$ by using the logistic distribution as an approximation to the discrete Binomial or Poisson distributions.

So we will approximate an optimal order quantity using the logistic distribution.

The probability density function of logistic distribution is

$$f(D)=\frac{m\,e^{\frac{-m}{\sigma}(D-\mu)}}{\sigma\,(1+e^{\frac{-m}{\sigma}(D-\mu)})^2}$$

Such that $m=\dfrac{\pi}{\sqrt{3}}\approx 1.8$

Using equation (2.4) to find an approximate order quantity:

$$\int_0^Q \frac{m\,e^{\frac{-m}{\sigma}(D-\mu)}}{\sigma\,(1+e^{\frac{-m}{\sigma}(D-\mu)})^2}dD=\frac{g}{g+h} \qquad (2.7)$$

Let $u=1+e^{\frac{-m}{\sigma}(D-\mu)} \rightarrow du=\dfrac{-m}{\sigma}e^{\frac{-m(D-\mu)}{\sigma}}$

Then $\int \frac{m\, e^{\frac{-m}{\sigma}(D-\mu)}}{\sigma\,(1+e^{\frac{-m}{\sigma}(D-\mu)})^2} = -\int u^{-2} du$

So that $\int_0^Q \frac{m\, e^{\frac{-m}{\sigma}(D-\mu)}}{\sigma\,(1+e^{\frac{-m}{\sigma}(D-\mu)})^2} dD = \left. \frac{1}{1+e^{\frac{-m}{\sigma}(D-\mu)}} \right|_0^Q$

$$= \frac{1}{1+e^{\frac{-m}{\sigma}(Q-\mu)}} - \frac{1}{1+e^{\frac{m}{\sigma}(\mu)}}$$

Substitute it into (2.4):

$$\frac{1}{1+e^{\frac{-m}{\sigma}(Q-\mu)}} - \frac{1}{1+e^{\frac{m}{\sigma}(\mu)}} = \frac{g}{g+h}$$

Solve the last equation for Q we get:

$$Q^* \approx \mu - \frac{\sigma}{m} \ln\left(\frac{h e^{m\mu/\sigma} - g}{g\left(2+e^{m\mu/\sigma}\right)+h}\right) \qquad (2.8)$$

Example: Consider a Poisson distribution with $\lambda = 4, h = 100, g = 1000$.

Using excel program and look for Q to find Q^* such that $F(Q^*) \geq \frac{g}{h+g}$

$$\sum_{D=0}^{Q} e^{-\lambda} \frac{\lambda^D}{D!} \geq \frac{1000}{100+1000}$$

And take the smallest Q that satisfies this condition.

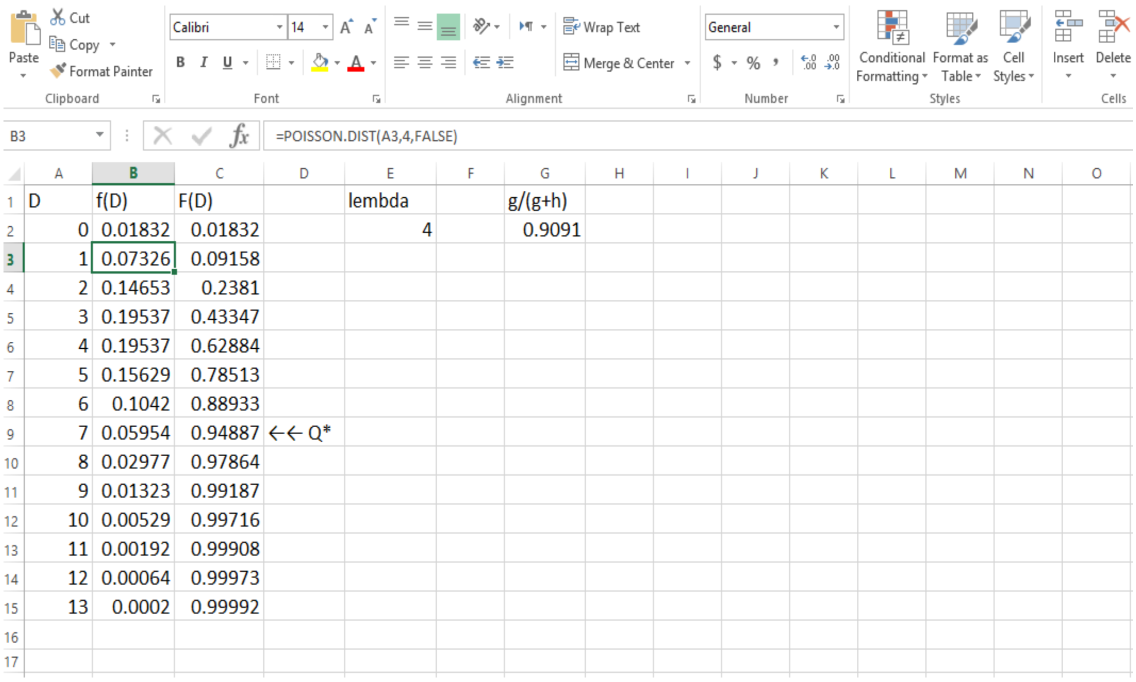
Figure 1 shows the Poisson probabilities $f(D)$ and the cumulative Poisson probabilities $F(D)$.

The optimal (maximum expected profit) value of Q can be found by finding the smallest value of Q such that $F(Q^*) \geq 0.9091$.

The optimal value of Q for this problem, therefore, is Q* = 7.

The cumulative Poisson distribution can be implemented in Excel with the function POISSON (Q, λ, TRUE) [16]. While Excel does not provide a function for the inverse of the cumulative Poisson, it is easy to find Q that satisfies equation (2.6) by using R-project and using the command "qpois (probability, lambda)" that returns the inverse of a Poisson-distribution function.

Using equation (2.8) we find an approximation value for Q^* and it is equal to 7 and the expected cost associated with this Q^* can be computed using equation (2.5) and it is equal to 6.9395 $.



B3 =POISSON.DIST(A3,4,FALSE)

D	f(D)	F(D)		lembda		g/(g+h)
0	0.01832	0.01832		4		0.9091
1	0.07326	0.09158				
2	0.14653	0.2381				
3	0.19537	0.43347				
4	0.19537	0.62884				
5	0.15629	0.78513				
6	0.1042	0.88933				
7	0.05954	0.94887	←← Q*			
8	0.02977	0.97864				
9	0.01323	0.99187				
10	0.00529	0.99716				
11	0.00192	0.99908				
12	0.00064	0.99973				
13	0.0002	0.99992				

Figure 1: Poisson probabilities with mean λ = 4.

Chapter Three

Parameter Estimation

Estimation is the process of finding an estimate, or approximation, which is a value that is usable for some purposes even if input data may be incomplete, uncertain, or unstable. The value is nonetheless usable because it is derived from the best information available. Typically, estimation involves "using the value of a statistic derived from a sample to estimate the value of a corresponding population parameter. The sample provides information that can be projected, to determine a range most likely to describe the missing information.

Note that an estimator is a function of the sample, while an estimate is the realized value of an estimator that is obtained when a sample is actually taken.

The quantity that we hope to guess is called the estimates [31].

- Types of Estimates:
  - Point estimate: single number that can be regarded as the most possible value of the parameter
  - Interval estimate: a range of numbers, called a confidence interval indicating, can be regarded as likely containing the true value of the parameter.

In this chapter we will clarify the point estimate and two methods of finding this type.

3.1 Point Estimate

The point estimation using particular functions of the data in order to estimate certain unknown of population parameter.

The goal of point estimation is to make a reasonable guess of the unknown value of a specified population quantity, e.g., the population mean.

Some Methods of finding point estimates:

1. Method of Moments
2. Maximum Likelihood
3. Bayes Estimators [21]

In the coming two sections we will clarify the last two methods of finding estimators.

3.1.1 Likelihood Function:

Let $f(\bar{Y} \mid \theta)$ denote the probability density function (PDF) that specifies the probability of observing data vector $\bar{Y}$ given the parameter θ.

Given a set of parameter values, the corresponding PDF will show that some data are more probable than other data.

In another case, we are faced with an inverse problem: Given the observed data and a model of interest, find the one PDF, among all the probability densities that the model prescribes, that is most likely to have produced the data.

To solve this inverse problem, we define the likelihood function by reversing the roles of the data vector $\bar{Y}$ and the parameter vector θ in $f(\bar{Y} \mid \theta)$, i.e. $L(\theta \mid \bar{Y}) = f(\bar{Y} \mid \theta)$

Such that $L(\theta \mid \bar{Y})$ represents the likelihood of the parameter θ given the observed data $\bar{Y}$; and as such is a function of θ . [12]

Example:

Given a binomial distribution with arbitrary values of p and n, such that the probability of a success on any trial, represented by the parameter p , and the number of trails , represented by n.

Suppose that the data y represents the number of successes in a sequence of n Bernoulli trials. So a general expression of the PDF of the binomial distribution is given by:

$$f(y \mid n, p) = \frac{n!}{y!(n-y)!} p^{y}(1-p)^{n-y} \quad , 0 \le p \le 1 \, ; y = 0,1....,n \qquad [34]$$

Which, as a function of y , specifies the probability of data y for a given parameters n and p

let $n = 9, p = 0.2$, The PDF in this case is given by:

$$f(y \mid n = 9, p = 0.2) = \frac{9!}{y!(9-y)!} 0.7^{y}(0.3)^{9-y} \quad , y = 0,1....,9$$

The shape of this PDF is shown in Figure 1:

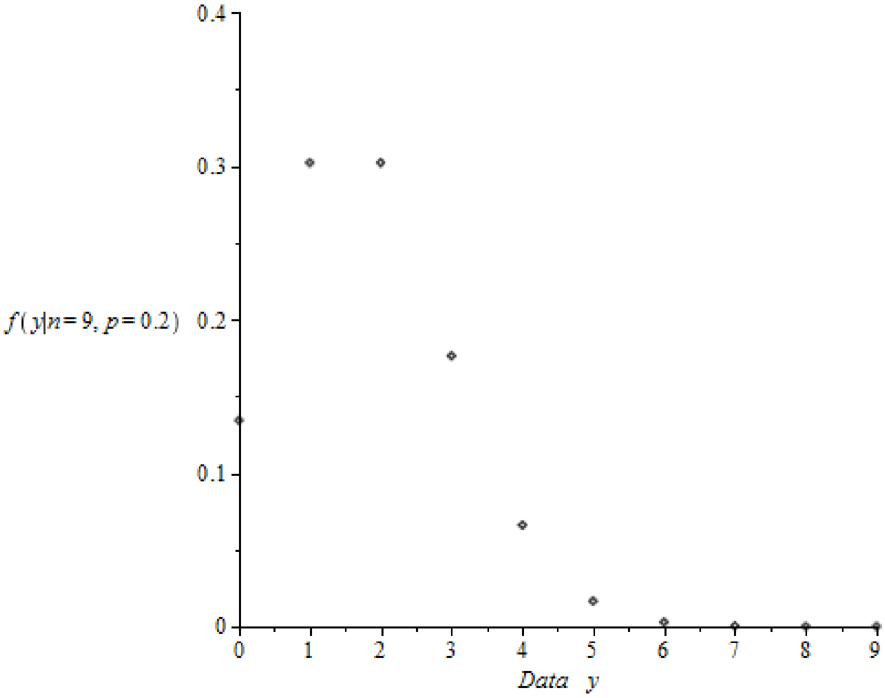


figure 1

For the likelihood function for y =6 and n = 9 is given by :

$$L(p \mid y=6, n=9) = \frac{9!}{6!(3)!} p^6 (1-p)^3 \quad , 0 \le p \le 1$$

The shape of this likelihood function is shown in Figure 2.

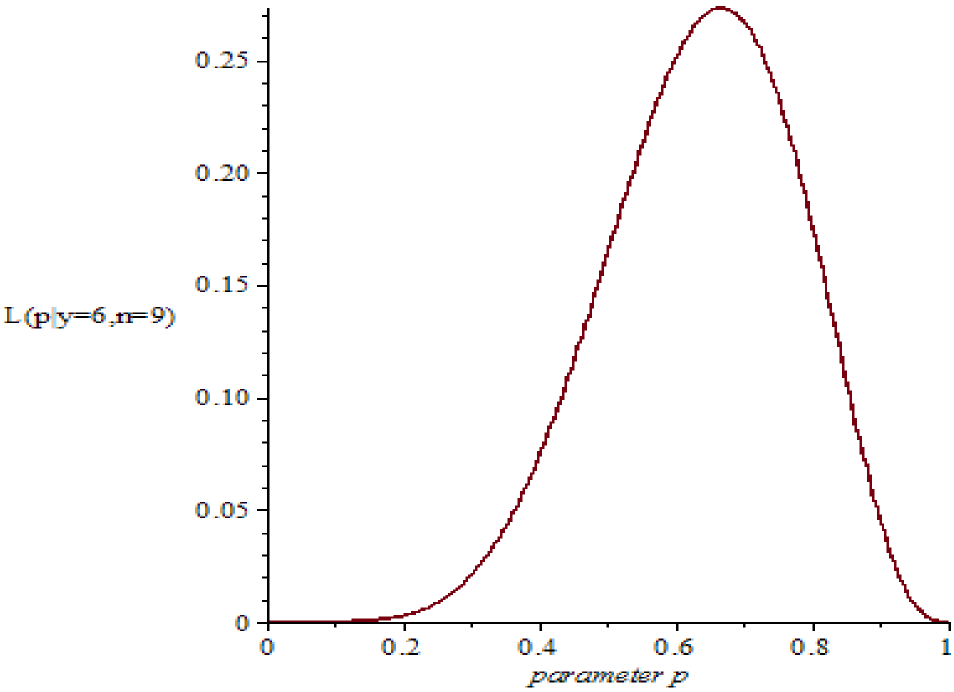


figure 2

There is an important difference between the PDF and the likelihood function, the two functions are defined on different axes, and therefore are not directly comparable to each other. Specifically, the PDF tells us the probability of a particular data value for a fixed parameter, whereas likelihood function tells us the likelihood of a particular parameter value for a fixed data set.

Note that the likelihood function in this figure is a curve because there is only one parameter beside n; which is assumed to be known. If the model has two parameters, the likelihood function will be a surface sitting above the parameter space[12].

Maximum Likelihood Estimators (MLE)

The principle of maximum likelihood estimation (MLE), originally developed by R.A. Fisher in the 1920s, states that the desired probability distribution is the one that makes the observed data ‘‘most likely,’’ which means that one must seek the value of the parameter vector that maximizes the likelihood function .The resulting parameter vector is called the MLE estimate.

Consider an experiment in which $(x_1, x_2, ...x_n)$ are independent and identically distributed (iid) random variables sample from a population with pdf or pmf $f(x \mid \theta_1, \theta_2 ......\theta_k)$, the likelihood function is defined by :

$$L(\theta \mid X) = L(\theta_1, \theta_2 ......\theta_k \mid x_1, x_2, ...x_n) = \prod_{i=1}^{n} f(x_i \mid \theta_1, \theta_2 ......\theta_k) \quad . \quad (3.1)$$

[21]

Definition:

For each sample x, let $\hat{\theta}(x)$ be parameter value at which $L(\theta \mid x)$ attains its maximum as a function of theta which x held fixed.

A maximum likelihood estimator of the parameter θ based on a sample X is $\hat{\theta}(X)$ [21].

The maximum likelihood estimator MLE, denoted by $\hat{\theta}(X)$, is the value of θ that maximizes $L(\theta)$.

The maximum of $\log(L(\theta))$ occurs at the same place as the maximum of $L(\theta)$ so maximizing the log-likelihood leads to the same answer as maximizing the likelihood function [23]. Often, it is easier to work with the log-likelihood.

Remark

If we multiply $L(\theta)$ by any positive constant (not depending on θ) then this will not change the MLE. Hence, we shall often be sloppy about dropping constants in the likelihood function [23].

Furthermore, if the sample is large, the method will typically yield an excellent estimator of θ.

Now we want to find θ such that $\log(L(\theta))$ is maximized, to do this we can use one of the following methods:

1. Graphically.
2. Optimization methodology.
3. Numerically.

In our work we will use the second method i.e. take the derivative of $\log(L(\theta))$ and find out the points θ where it's zero: $(\log(L(\theta)))' = 0$

$(\log(L(\theta)))'$ is the slope at θ. If its zero it means that you have found either a minimum, a maximum or a saddle point. We are not interested in saddle points so we want to check that the points where $(\log(L(\theta)))'$ is zero also have the $(\log(L(\theta)))''$ is non-zero. $(\log(L(\theta)))''$ gives a measure for the "curvature" of $\log(L(\theta))$ at that point. Saddle points are horizontal hence have $(\log(L(\theta)))''$ equal to zero.

3.1.2 MLE for a Poisson distribution:

Let $(x_1, x_2, \ldots\ldots, x_n)$ are the samples taken from Poisson distribution, and the probability mass function is given by:

$$f(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , \quad \lambda \text{ is unknown.}$$

So the likelihood function is given by:

$$L(\lambda) = f(x_1,\lambda)\cdot f(x_2,\lambda)\cdot \ldots\ldots f(x_n,\lambda)$$

$$= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!}\cdot\frac{e^{-\lambda}\lambda^{x_2}}{x_2!}\cdot\ldots\ldots\frac{e^{-\lambda}\lambda^{x_n}}{x_n!}$$

$$\rightarrow L(\lambda) = e^{-n\lambda}\cdot\frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \quad (3.2)$$

Then we find the natural logarithm likelihood function:

$$\ln(L(\lambda)) = \ln\left(\frac{1}{\prod_{i=1}^{n} x_i!}\right) + \ln(e^{-n\lambda}) + \ln\left(\lambda^{\sum_{i=1}^{n} x_i}\right)$$

$$= -\sum_{i=1}^{n} \ln x_i - n \cdot \lambda + \sum_{i=1}^{n} x_i (\ln \lambda)$$

In order to find the maximum λ , take the derivatives of the last expression with respect to λ and equate it to zero.

$$\frac{\partial \ln(L(\lambda))}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

$$0 \qquad = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

$$n = \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

$$\Rightarrow \hat{\lambda} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \qquad (3.3)$$

I.e. $\hat{\lambda}$ is equal to the mathematical mean of the sample $\hat{\lambda} = \overline{x}$

Thus the mean of the sample gives the maximum likelihood estimation of the parameter λ .

3.1.3 MLE for a Binomial distribution:

Let $\overline{X}$ be a random variable with parameter p . Let $(x_1, x_2, \ldots\ldots, x_m)$ be the independent random samples of $\overline{X}$.

Recall the probability mass function for the binomial distribution with parameter p is:

$$f(x,p)=\binom{n}{x}\cdot p^{x}\cdot(1-p)^{n-x},x=0,1,\ldots,n$$

Then the likelihood function of the sample is:

$$L(p)=f(x_1,p)\cdot f(x_2,p)\cdot\ldots\ldots\cdot f(x_m,p)$$

$$=\prod_{i=1}^{m}\binom{n}{x_i}\cdot p^{x_i}\cdot(1-p)^{n-x_i} \quad (3.4)$$

Taking the natural logarithm on both sides:

$$\ln\left(L(p)\right)=\ln\left(\prod_{i=1}^{m}\binom{n}{x_i}\cdot p^{x_i}\cdot(1-p)^{n-x_i}\right)$$

$$=\sum_{i=1}^{m}\left[\ln\binom{n}{x_i}+x_i\cdot\ln p+(n-x_i)\cdot\ln(1-p)\right]$$

$$=\sum_{i=1}^{m}\ln\binom{n}{x_i}+\left(\sum_{i=1}^{m}x_i\right)\cdot\ln p+\left(m\cdot n-\sum_{i=1}^{m}x_i\right)\cdot\ln(1-p)$$

Since $\ln(L(p))$ is a continuous function of p, then it has a maximum value. Now we will take the derivatives of the last expression with respect to p and setting it equal to zero, so:

$$\frac{\partial\ln(L(p))}{\partial p}=0+\frac{1}{p}\cdot\sum_{i=1}^{m}x_i-\frac{1}{1-p}\cdot\left(m\cdot n-\sum_{i=1}^{m}x_i\right)$$

$$0=\frac{1}{p}\cdot\sum_{i=1}^{m}x_i-\frac{1}{1-p}\cdot\left(m\cdot n-\sum_{i=1}^{m}x_i\right)$$

$$0=(1-p)\cdot\sum_{i=1}^{m}x_i-p\cdot(m\cdot n-\sum_{i=1}^{m}x_i)$$

$$\sum_{i=1}^{m}x_i=p\cdot m\cdot n$$

$$\Rightarrow \hat{p}=\frac{1}{m\cdot n}\cdot\sum_{i=1}^{m}x_i \quad (3.5)$$

3.1.4 MLE for an Exponential distribution:

Let $(x_1, x_2, \ldots\ldots, x_n)$ be a random sample taken from exponential distribution, and the probability density function given by:

$$f(x,\lambda)=\lambda\cdot e^{-\lambda\cdot x}$$

The likelihood function of the sample is given by:

$$L(\lambda)=\prod_{i=1}^{n}\lambda\cdot e^{-\lambda\cdot x_i}$$

$$L(\lambda)=\lambda^n\cdot e^{-\lambda\cdot\sum_{i=1}^{n}x_i} \qquad (3.6)$$

Taking the natural logarithm on both sides:

$$\ln(L(\lambda))=n\cdot\ln\lambda+\ln e^{-\lambda\cdot\sum_{i=1}^{n}x_i}$$

$$=n\cdot\ln\lambda-\lambda\cdot\sum_{i=1}^{n}x_i$$

In order to find the maximum λ, take the derivatives of the last expression with respect to λ and equate it to zero.

$$\frac{\partial \ln(L(\lambda))}{\partial \lambda} = n \cdot \frac{1}{\lambda} - \sum_{i=1}^{n} x_i$$

$$0 = n \cdot \frac{1}{\lambda} - \sum_{i=1}^{n} x_i$$

$$n \cdot \frac{1}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} \quad (3.7)$$

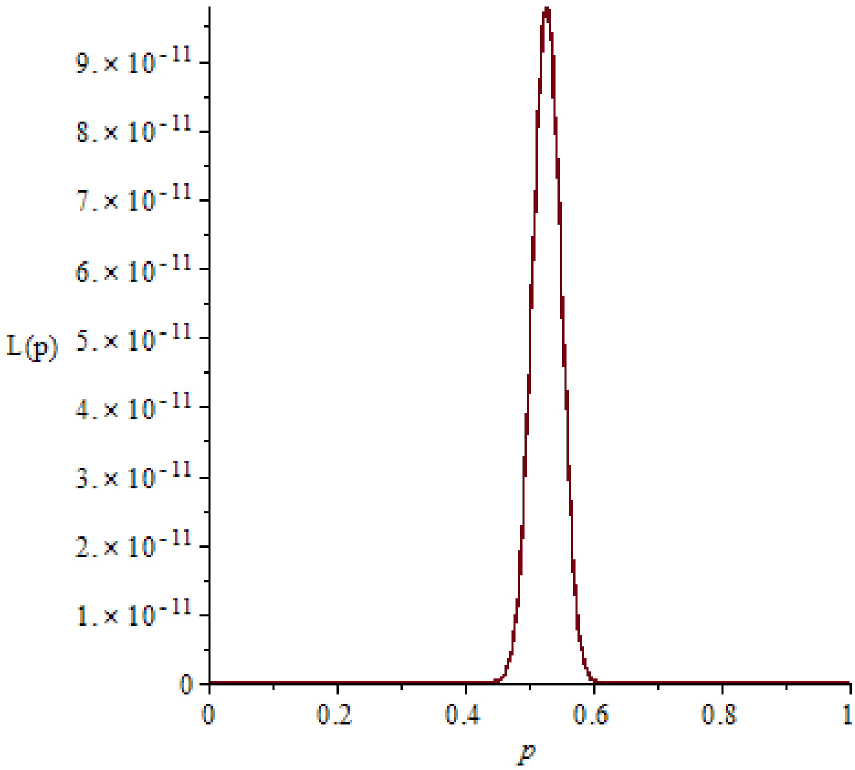
Thus the maximum likelihood estimator of λ is equal to the inverse of the mean of the sample.

Example:

The owner of the news stand pays 1 $ for a copy of the newspaper and sells it for 8$. Newspapers left at the end of the day are recycled for an income of 4$ a copy. Assume the newsvendor has a pool of 50 customers that come every day to the stand .Each customer may buy a newspaper with probability p. It is a well-known fact that any experiment comprising a sequence of n (n=50 in our example) Bernoulli trials, each having the same “yes” probability p and the same “No” probability 1 − p , can be represented by a random variable $bin(n, p)$ that follows a binomial distribution. Assume the probability of success p is not known, and the owner of the stand wants to determine the optimal number of newspapers that must to be stocked at the begging of the day. So in order to find the optimal quantity we can use equation (2.6) but p is not known, so we need to estimate it using MLE as discussed later.

Assume we have a set of past demand sample {28, 28, 27, 24, 25, 26, 28, 28, 23, 27} consider a newsvendor. So, as mentioned in subsection (3.1.3) we can find an estimation of p by using equation (3.5) we have

$$\hat{p} = \frac{264}{10 \cdot 50} \rightarrow \hat{p} = 0.528$$



likelihood function for Binomial with n = 50 and $\sum_{i=1}^{10} x_i = 264$. *The MLE is approximatly equal* = 0.5

Now, we can find the optimal quantity as clarified in subsection (2.3.2) from our example, unit overage cost =5-4=1\$ and the unit underage cost = 8-5=3\$. Thus the optimal order quantity is equal to 29 and the expected total cost is equal to 4.4615 \$.

3.1.5 Bayes estimators:

The Bayesian approach to statistics is fundamentally different from the classical approach that we have been discussing.

The main features of Bayesian approach is that parameters are random variables with probabilities, also we can make probability statements about parameters, even though they are fixed constants.

We make inferences about a parameter, by producing a probability distribution for the parameter. Then we can infer the value of the parameter such as point estimates and interval estimates may then be extracted from this distribution [23].

We will discuss the Bayesian approach in statistics.

A random sample $X_1,...,X_n$ is drown from a population indexed by θ.

θ,in Bayesian approach, is considered to be a quantity whose variation can be described by a probability distribution (called the prior distribution).This is a subjective distribution , based on the experimenter's belief , and is formulated before the data are seen. A sample is then taken from a population indexed by θ and the prior distribution is updated with this sample information. The updated prior is called the posterior distribution. This updating is done with the use of Bayes' Rule [21].

Bayesian analysis can be outlined in the following steps.

1. Formulate a probability model for the data. If the n data values to be observed are $x_1,\ldots,x_n$, and the unknown parameter is denoted θ, then, assuming that the observations are made independently, we are interested in choosing a probability function $f(x_i \mid \theta)$ for the data.

2. Decide on a prior distribution, which quantifies the uncertainty in the values of the unknown model parameters before the data are observed. The prior distribution can be viewed as representing the current state of knowledge, or current description of uncertainty, about the model parameters prior to data being observed.
3. Observe the data, and construct the likelihood function based on the data and the probability model formulated in step 1. The likelihood is then combined with the prior distribution from step 2 to determine the posterior distribution, which quantifies the uncertainty in the values of the unknown model parameters after the data are observed.
4. Summarize important features of the posterior distribution, or calculate quantities of interest based on the posterior distribution. These quantities constitute statistical outputs, such as point estimates [29].

To obtain the posterior distribution, $f(\theta|X)$, the probability distribution of the parameters once the data have been observed, we apply Bayes' theorem:

$$f(\theta| X) = \frac{f(X \mid \theta) f(\theta)}{\int f(X \mid \theta) f(\theta) d\theta} \qquad (3.8)$$

Since we have n iid observation we replace $f(X \mid \theta)$ with $L(\theta \mid X) = \prod_{i=1}^{n} f(x_i \mid \theta)$ then:

$$f(\theta| X) = \frac{L(\theta \mid X) f(\theta)}{\int L(\theta \mid x) f(\theta) d\theta} \propto L(\theta \mid X) f(\theta) \qquad (3.9)$$

In the right hand side of the last equation, we threw away the denominator $\int L(\theta \mid x) f(\theta) d\theta$ which is a constant that does not depend on θ ; this quantity call the normalizing constant.

We can summarize all this by writing:

'Posterior is proportional to likelihood times prior' [23].

To get actual posterior we will multiply the prior distribution by the likelihood, and then determine the normalizing constant that forces the expression to integrate to 1 to make sure it is a probability distribution.

The posterior distribution summarizes our belief about the parameter after seeing the data. It takes into account our prior belief and the data (likelihood). A graph of the posterior shows us all we can know about the parameter. A distribution is hard to interpret. Often we want to find a few numbers that characterize it. These include measures of location that determine where most of the probability is on the number line, and measures of spread that determine how widely the probability is spread. [4]

3.1.6 Binomial Bayes Estimation:

Let $x_1, \ldots, x_m$ be iid $binom(n, p)$ and $y = \sum_{i=1}^{m} x_i$, assume the prior distribution on p is $beta(\alpha, \beta)$

So the prior distribution is:

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \qquad (3.10)$$

Let $p_\circ$ be our prior mean for the proportion, and let $\delta_\circ$ be our prior standard deviation for the proportion. The mean of $beta(\alpha,\beta)$ is $\frac{\alpha}{\alpha+\beta}$ set this equal to what our prior belief about the mean , also the standard deviation of the $beta(\alpha,\beta)$ is $\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$ set this equal to what our prior belief about the standard deviation [4]and then we can find the two parameters α and β.

Also the likelihood function is given by

$$L(p)=\prod_{i=1}^{m}\binom{n}{x_i}\cdot p^{x_i}\cdot(1-p)^{n-x_i}$$

So, the posterior distribution is proportional to the product of the Beta prior distribution and the likelihood function

$$f(p|X)\propto L(p|X)f(p)=\prod_{i=1}^{m}\binom{n}{x_i}\cdot p^{x_i}\cdot(1-p)^{n-x_i}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}$$

(3.11)

To get the actual posterior we need to divide the last expression by the normalizing constant:

The normalizing constant

$$=\int_0^1\left(\prod_{i=1}^{m}\binom{n}{x_i}\cdot p^{x_i}\cdot(1-p)^{n-x_i}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}\right)dp$$

Then the posterior distribution is:

$$f(p|X)=\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+y)\Gamma(\beta+n-y)}p^{\alpha+y-1}(1-p)^{\beta+n-y-1} \quad (3.12)$$

I.e. the posterior distribution is equal to the beta function with parameters $\alpha + y$ and $\beta + n - y$

Example :

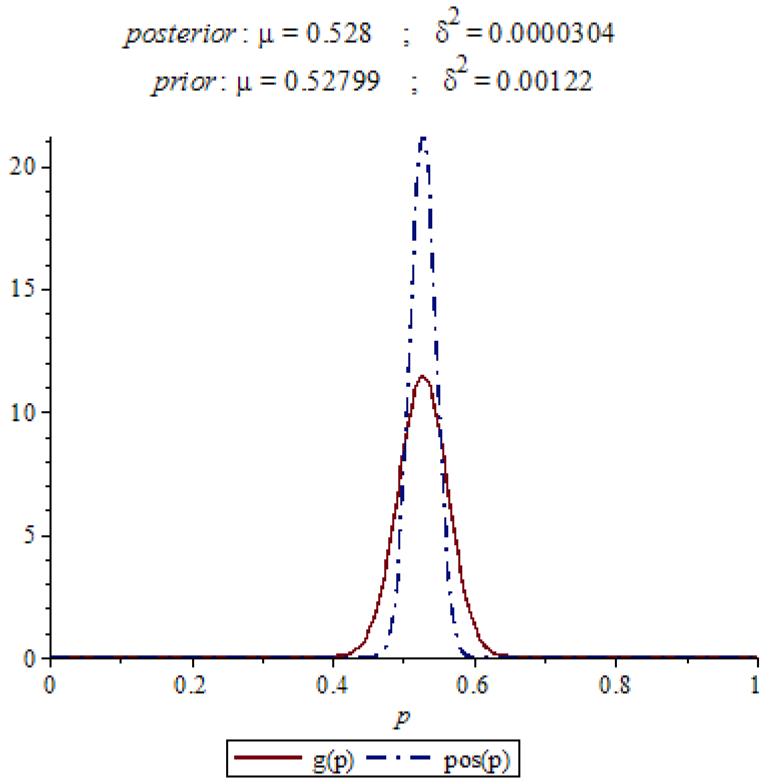
In the previous example if we need to estimate the parameter p using Bayesian approach.

Then y = 264 and our prior distribution is $beta(\alpha, \beta)$ .Set prior mean = 0.528 and prior variance = 0.001216, then α = 107.6842 and β =96.2632.

Then the posterior distribution is $beta(107.6842 + 264, 96.2632 + 500 - 264)$

And we can estimate the parameter p by the mean of the posterior distribution:

$\hat{p}$ =0.528.



3.1.7 Poisson Bayes Estimation:

Let $x_1,....,x_m$ be iid $Poisson(\lambda)$ and $y=\sum_{i=1}^{m} x_i$, assume the prior distribution on λ is $Gamma(\alpha,\beta)$

So the prior distribution is:

$$f(\lambda)=\frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)} \quad (3.13)$$

Let $\mu_{\circ}$ be our prior mean, and let $\delta_{\circ}$ be our prior standard deviation. The mean of $Gamma(\alpha,\beta)$ is $\frac{\alpha}{\beta}$ set this equal to what our prior belief about the mean , also the standard deviation of the $Gamma(\alpha,\beta)$ is $\frac{\alpha}{\beta^2}$ set this equal to what our prior belief about the standard deviation. And then we can find the two parameters α and β .

Also the likelihood function is given by

$$L(\lambda)=\prod_{i=1}^{m}\frac{e^{-\lambda}\cdot\lambda^{x_i}}{x_i\,!}$$

So, the posterior distribution is proportional to the product of the Gamma prior distribution and the likelihood function

$$f(\lambda|\,X)\propto L(\lambda\,|\,X)\,f(\lambda)=\prod_{i=1}^{m}\frac{e^{-\lambda}\cdot\lambda^{x_i}}{x_i\,!}\frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)} \quad (3.14)$$

To get the actual posterior we need to divide the last expression by the normalizing constant:

The normalizing constant $=\int_{0}^{\infty}(\prod_{i=1}^{m}\frac{e^{-\lambda}\cdot\lambda^{x_i}}{x_i\,!}\frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)})d\lambda$

Then the posterior distribution is:

$$f(\lambda | X) = \frac{(\beta+n)^{\alpha+y} \lambda^{\alpha+y-1} e^{-\lambda(\beta+n)}}{\Gamma(\alpha+y)} \quad (3.15)$$

I.e. the posterior distribution is equal to the Gamma function with parameters $\alpha + y$ and $\beta + n$.

3.1.8 Exponential Bayes Estimation:

Let $x_1, ..., x_m$ be iid $Exp(\lambda)$ and $y = \sum_{i=1}^{m} x_i$, assume the prior distribution on λ is $Gamma(\alpha, \beta)$

So the prior distribution is:

$$f(\lambda) = \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)} \quad (3.16)$$

Let $\mu_{\circ}$ be our prior mean, and let $\delta_{\circ}$ be our prior standard deviation. The mean of $Gamma(\alpha, \beta)$ is $\frac{\alpha}{\beta}$ set this equal to what to what our prior belief about the mean , also the standard deviation of the $Gamma(\alpha, \beta)$ is $\frac{\alpha}{\beta^2}$ set this equal to what our prior belief about the standard deviation. And then we can find the two parameters α and β .

Also the likelihood function is given by

$$L(\lambda) = \prod_{i=1}^{m} \lambda e^{-\lambda x_i}$$

So, the posterior distribution is proportional to the product of the Beta prior distribution and the likelihood function

$$f(\lambda | X) \propto L(\lambda | X) f(\lambda) = \prod_{i=1}^{m} \lambda e^{-\lambda x_i} \cdot \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)}$$

To get the actual posterior we need to divide the last expression by the normalizing constant:

The normalizing constant $= \int_0^\infty (\prod_{i=1}^{m} \lambda e^{-\lambda x_i} \cdot \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)}) d\lambda$

Then the posterior distribution is:

$$f(\lambda | X) = \frac{(\beta + y)^{\alpha+n} \lambda^{\alpha+n-1} e^{-\lambda(\beta+y)}}{\Gamma(\alpha+n)} \quad (3.17)$$

I.e. the posterior distribution is equal to the Gamma function with parameters $\alpha + n$ and $\beta + y$.

3.2 Confidence interval:

When we wish to estimate an unknown parameter θ confidence intervals provide a method of adding more information to an estimator $\hat{\theta}$.

As we discussed in point estimation, when we need to estimate the value of an unknown parameter from a random sample, we have a single estimate, and we have no indication of just how good our best estimate is, also a single estimate has always, however, been realized that this single value is of little use unless associated with a measure of its reliability, but it was neither easy to give any precise definition of this measure of probability nor to assess the extent of error involved in estimating the value of the parameter from the sample [8], so that statisticians have cleverly developed another type of estimate. This new type estimate, called a confidence interval or interval estimate, consists of a range (or an interval) of values instead of just a single value. [28]

In other words a confidence interval for a population parameter consists of a range of values, restricted by a lower and an upper limit.

The lower and upper bounds of a confidence interval are random (they may change from sample to sample). In a given sample, however, they are known numbers.

A confidence level, (1-α)%, refers to the percentage of all possible samples that can be expected to include the true population parameter. For example, suppose all possible samples were selected from the same population, and a confidence interval were computed for each sample. A 95% confidence level implies that 95% of the confidence intervals would include the true population parameter.

3.2.1. Confidence interval for the Binomial distribution

Description: Let x be the number of successes in a random sample of size m. A success is observed if y_i has a specific characteristic; such that $y_i \in \{y_1, y_2, ..., y_m\}$ and a failure is observed if y_i does not have that characteristic. The point estimation of the parameter p is equal to $\frac{x}{n \cdot m}$ (as discussed in section 3.1.3)

There are several ways to construct a confidence interval for the parameter p for example:

Wilson's score interval (Wilson, 1927),

The Wald interval (Wald & Walfowitz, 1939),

The adjusted Wald interval (Agresti & Coull, 1998),

And the 'exact' Clopper-Pearson interval (Clopper & Pearson, 1934). [5]

In our work we will focus on the Wald interval Method and the Clopper Pearson Method.

Clopper Pearson method

Clopper-Pearson method is based on the exact binomial distribution, some authors refer to this as the "exact" procedure because of its derivation from the binomial distribution. If $X = x$ is observed, then the Clopper–Pearson interval is defined by (p_{lb}, p_{ub})

Where p_{lb}, p_{ub} are, respectively, the solution in p to the equations:

$$p(X \geq x) = \alpha / 2 \quad (3.18)$$

And $p(X \leq x) = \alpha / 2$ (3.19) [1]

As discussed in [11] the computation of (p_{lb}, p_{ub}) is simplified by using quantiles from the beta distribution. Let $f(t, \alpha, \beta)$ be the density function of a $Beta\ (\alpha, \beta)$ random variable. Then

$$p(X \geq x) = \int_0^p f(t, x, n - x + 1)\, dt \quad (3.20)$$

When (3.20) is plugged into (3.18) and (3.19), the problem of finding (p_{lb}, p_{ub}) reduces to inverting the distribution functions of two beta distributions. So the lower endpoint is the $\alpha / 2$ quantile of a beta distribution, $Beta\ (x, n - x + 1)$, and the upper endpoint is the $1 - \alpha / 2$ quantile of a beta distribution, $Beta\ (x + 1, n - x)$

Consequently, the endpoints of the Clopper–Pearson interval are given by quantiles of beta distributions:

$$(p_{lb}, p_{ub}) = (Beta(\alpha/2, x, n-x+1), Beta(1-\alpha/2, x+1, n-x))$$

(3.21)

When X is neither 0 nor n, closed-form expressions for the interval bounds are available.

But when X = 0 the interval is $(0, 1-(\alpha/2)^{1/n})$ and when X = n it is $((\alpha/2)^{1/n}, 1)$. For other values of X, (3.21) must be evaluated numerically.[11]

Furthermore, this interval can also expressed using quantiles from the F distributions based on the relationship between the binomial distribution and the F distribution as follows:

$$\frac{1}{1+\frac{n-x+1}{x}F_{2(n-x+1),2x,\alpha/2}} \le p \le \frac{\frac{x+1}{n-x}F_{2(x+1),2(n-x),\alpha/2}}{1+\frac{x+1}{n-x}F_{2(x+1),2(n-x),\alpha/2}} \qquad (3.22)$$

Where $F_{v_1,v_2,\alpha}$ is the upper $100\times(1-\alpha)^{th}$ percentile from a F distribution with v_1 and v_2 degrees of freedom [14].

Wald interval method

The normal theory approximation of a confidence interval for a proportion is known as the Wald interval [5].

Normal approximation method is good and easy to compute estimate of the Binomial distribution.

As discussed in [5] the formula used to derive the confidence interval using the normal approximation is

$$(p_{lb}, p_{ub}) = (\hat{p} - z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}\ ,\ \hat{p} + z_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})/n}\ ) \quad (3.23)$$

Where $z_{\alpha/2}$ is the $\alpha/2$ critical value from a standard normal distribution and $\hat{p}$ is a point estimation of the parameter p using MLE as discussed in section (3.1.3)

The Wald interval suffers from particularly erratic coverage properties, and cannot be recommended for general use

Normal approximation method works well when n is large, and p is neither very small nor very large. But for very small values of p it doesn't provide accurate results. Due to the inaccuracy of the normal approximation method, many statisticians started using the exact Clopper-Pearson method.[14]

The confidence interval may be used if:

1. np, n(1 − p)are ≥ 5 (or 10);

2. np (1 − p)≥ 5 (or 10) [24].

3.2.2 Confidence interval for the Poisson distribution

Let $y_1, y_2, ..., y_n$ be a random sample from $Poisson(\lambda)$. Let $x = \sum_{i=1}^{n} y_i \sim Poisson(n\lambda)$, the classic method of constructing exact confidence intervals for parameter λ of Poisson distribution is to use the fiducial interval (λ_l, λ_u) such that λ_l *and* λ_u are, respectively, the solutions in λ to the equations:

$$\sum_{i=x}^{\infty} \frac{e^{-n\cdot\lambda_l} n \cdot \lambda_l^{i}}{i!} = \alpha/2 \quad (3.24)$$

$$\sum_{i=0}^{x} \frac{e^{-n\cdot\lambda_u} n\cdot\lambda_u{}^i}{i!} = \alpha/2 \quad (3.25)\ [20]$$

The main problem with using this method is the difficulty in computing the cumulative Poisson probability expressions [15]. So that instead of evaluating Poisson cumulative probabilities in (3.24) and (3.25), as discussed in [19] one can use the relationship between the Poisson and the chi-squared distributions:

$$\sum_{i=x}^{\infty} \frac{e^{-n\cdot\lambda} n\cdot\lambda^i}{i!} = p(\chi_{2x}{}^2 \le 2n\lambda)$$

Then the confidence interval of the Poisson distribution can be expressed as:

$$(\lambda_l, \lambda_u) = \left(\frac{1}{2n} \chi^2_{2x,1-\alpha/2}, \frac{1}{2n} \chi^2_{2(x+1),\alpha/2} \right) \quad (3.26)$$

Where $x = \sum_{i=1}^{n} y_i$ and squared -quantile of the chi v^{th} denotes the $\chi^2_{v,\alpha}$ distribution with degree of freedom = α and where we define $\chi^2_{0,\alpha} \equiv 0$ [20].

3.2.3 Confidence interval for the Exponential distribution

We will use the exact confidence interval for the exponential distribution as discussed in [38].

Suppose $X_1, X_2, ..., X_n$ are independent exponential random variables each having exponential distribution with parameter λ , let $y = \sum_{i=1}^{n} X_i$ then y has a gamma random variable with parameters n and $\frac{1}{\lambda}$ [7].

So a $100(1-\alpha)$ percent confidence interval for λ is (λ_l, λ_u) such that λ_l *and* λ_u are, respectively, the solutions in λ to the equations:

$$\int_{y}^{\infty}\left(\frac{(\frac{1}{\lambda})^{n} t^{n-1} e^{-t\frac{1}{\lambda}}}{\Gamma(n)}\right) dt = \frac{\alpha}{2}$$

$$\int_{0}^{y}\left(\frac{(\frac{1}{\lambda})^{n} t^{n-1} e^{-t\frac{1}{\lambda}}}{\Gamma(n)}\right) dt = \frac{\alpha}{2}$$

And it can be expressed using the quantiles from the chi-square distribution:

$$(\lambda_l, \lambda_u) = \left(\frac{\chi^2_{\alpha/2,2n}}{2\sum_{i=1}^{n} X_i}\ ,\ \frac{\chi^2_{1-\alpha/2,2n}}{2\sum_{i=1}^{n} X_i}\right) \quad (3.27) \quad [38]$$

Such that: n is the number of observations and quantile v^{th} denotes the $\chi^2_{v,\alpha}$ of the chi-squared distribution with degree of freedom α and where we define $\chi^2_{0,\alpha} \equiv 0$

Chapter Four

Combine confidence interval analysis and inventory optimization

Consider the single period inventory control problem with a single item, we will take the situation in which the manager knows the type of the random demand distribution, but he doesn't know the value of some parameter of this distribution. Fortunately, the manager have a set of M past realizations of the demand. Under these partial realizations we will compute estimation of the unknown parameter and depending on this estimation we will find a range of order quantities, and this range will include-under confidence coefficient $1-\alpha$ -the optimal order quantity, and then we will compute an interval for the expected total cost associated with the range of order quantities.

4.1 Binomial demand

In this section we will consider the situation where the demand has a Binomial distribution with two parameters n and p ,($binom(n, p)$). In the first case all of its parameters are known, as in the previous discussion, we can directly find the optimal order quantity and the expected total cost that the manager will infer.

But in the other case where the parameter p (probability of success) is not known, and we have a set of past demand samples, we need to use this set in order to estimate the parameter p .

As discussed in the previous chapters, in order to estimate any parameter we can use the point estimation or the interval estimation.

Combine confidence interval analysis and inventory optimization Since we have a set of past demand sample, we will use it to estimate the parameter p by constructing a confidence interval.

Let $y_1,....y_m$ are the sample of a past demand for m-days, using this data to compute a lower and upper bounds of the confidence interval for the probability of success in the binomial demand.

We will construct an exact confidence interval for the unknown parameter with a confidence coefficient $(1-\alpha)$:

Since $y_i \sim binom(n,p)$ so

$$x = \sum_{i=1}^{m} y_i \sim binom(n \cdot m, p)$$

The bounds of the confidence interval for the probability of success p (p_{lb}, p_{ub}) are, respectively, the solution in p to the following two equations:

$$\sum_{i=x}^{n\cdot m} \binom{n\cdot m}{i} p_{lb}^{i} (1-p_{lb})^{n\cdot m-i} = \alpha/2 \qquad (4.1)$$

$$\sum_{i=0}^{x} \binom{n\cdot m}{i} p_{u}^{i} (1-p_{u})^{n\cdot m-i} = \alpha/2 \qquad (4.2)$$

Again, we can express this interval using quantiles from the beta distribution as we discussed in section (3.2.1):

$$(beta(\alpha/2, x, n\cdot m - x + 1), beta(1-\alpha/2, x+1, n\cdot m - x)) \qquad (4.3)$$

After we construct an interval for the parameter p , we will now determine a set of quantities that contains the optimal order quantity with confidence coefficient $(1-\alpha)$.

Let Q^*_{lb} be an optimal order quantity of the single period inventory under binomial demand $binom(n, p_{lb})$ with probability success p_{lb}.

And let Q^*_{ub} be an optimal order quantity of the single period inventory under binomial demand $binom(n, p_{ub})$ with probability success p_{ub}.

And we can find the values of Q^*_{lb} and Q^*_{ub} quantities as we clarify in section (2.3.2)

So after computing the lower and upper optimal quantities we get, with confidence coefficient $1-\alpha$, a set that contains the optimal order quantity Q^* i.e.

$$Q^* \in A = \{Q^*_{lb}, Q^*_{lb}+1, ..., Q^*_{ub}-1, Q^*_{ub}\}.$$

At this point the manager has a set of quantities that he can choice on member of this set to order it at the beginning of the day. But he needs an information about the cost he will pay.

In other words we need to compute an interval for the expected estimated cost associated with his choice and the expected estimated cost that the manager will face whatever the order quantity he chooses from the set.

We will now construct a confidence interval, with confidence coefficient $(1-\alpha)$, of the expected total cost that the manager will pay if he order Q quantity that he choices from the set A.

Recall the expected total cost associated with the order quantity Q under the binomial random demand, $binom(n,p)$, as we discussed in section (2.3.2) is:

$$E(G(Q)) = h\sum_{D=0}^{Q}(Q-D)\binom{n}{D}p^D(1-p)^{n-D} + g\sum_{D=Q+1}^{n}(D-Q)\binom{n}{D}p^D(1-p)^{n-D} \quad (4.4)$$

Such that h represents the unit overage cost and g represents the unit underage cost.

Consider the function:

$$G(p) = h\sum_{D=0}^{Q}(Q-D)\binom{n}{D}p^D(1-p)^{n-D} + g\sum_{D=Q+1}^{n}(D-Q)\binom{n}{D}p^D(1-p)^{n-D} \quad (4.5)$$

In which the order quantity Q is fixed and the probability of success p is a variable.

Proposition: $G(p)$ is a convex in the continuous parameter p

PROOF:

Firstly, we can rewrite the function (4.5) as:

$$G(p) = h(Q-np) + (g+h)\sum_{i=Q}^{n}(1-F(i)) \qquad (4.6)$$

In order to prove $G(p)$ is a convex we need to show $\frac{\partial^2 G(p)}{\partial p^2} \geq 0$ which is equivalent to $\frac{\partial^2}{\partial p^2}(\sum_{i=Q}^{n}(1-F(i))) \geq 0$ in other words we need to show $\frac{\partial^2}{\partial p^2}\sum_{i=Q}^{n}F(i) \leq 0$

We can rewrite $F(i) = F(i,n,p) = \sum_{k=0}^{i}\binom{n}{k}p^k(1-p)^{n-k}$ using the regularized incomplete beta function:

$$F(i,n,p) = (n-i)\binom{n}{i}\int_0^{1-p} s^{n-i-1}(1-s)^i\, ds$$

We will compute the first derivative of $F(i,n,p)$ using Leibniz's rule:

$$\begin{aligned}\frac{\partial}{\partial p}F(i,n,p) &= -n\binom{n-1}{i}(1-p)^{n-i-1}p^i \\ &= -n\, f(i, n-1, p) \\ &= -n(F(i,n-1,p) - F(i-1,n-1,p))\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial p^2}F(i,n,p) &= -n\left(\frac{\partial}{\partial p}F(i,n-1,p) - \frac{\partial}{\partial p}F(i-1,n-1,p)\right) \\ &= -n[-(n-1)(F(i,n-2,p) - F(i-1,n-2,p)) + (n-1)(F(i-1,n-2,p) - F(i-2,n-2,p))] \\ &= n(n-1)[f(i,n-2,p) - f(i-1,n-2,p)]\end{aligned}$$

$$\frac{\partial^2}{\partial p^2}\sum_{i=Q}^{n}F(i,n,p) = n(n-1)[f(Q,n-2,p) - f(Q-1,n-2,p) + f(Q+1,n-2,p) - f(Q-2,n-2,p) + \ldots + f(n,n-2,p) - f(n-1,n-2,p)]$$

All terms cancel out except $-f(Q-1,n-2,p)$ and $f(n,n-2,p)=0$ then:

$$\frac{\partial^2}{\partial p^2}\sum_{i=Q}^{n}F(i,n,p) = -n(n-1)f(Q-1,n-2,p)$$ which is less than zero.

So the cost function is a convex in the continuous parameter p.

Assume that we choose a quantity Q from the set A, we will try to find an upper and lower bound for the associated expected cost.

To do this we need to find p^* that minimize the cost function

Since $G(p)$ is convex in p then we can use unconstrained convex optimization approach in order to find p_l^Q that minimizes $G(p)$ also we can find p_u^Q that maximizes $G(p)$ over an interval (p_{lb}, p_{ub}).

We have now an interval $(G(p_l^Q), G(p_u^Q))$ that contains lower and upper bound for the expected cost that the manager will face if he orders Q quantity.

We will repeat this step for each $Q \in A$, and then we have an upper and lower bound for expected for every Q.

We can find the interval (c_{lb}, c_{ub}), with confidence coefficient $1-\alpha$, for the estimated cost that the manager will infer whatever quantity he orders from the set A by using the following formulas:

The lower bound is:

$$c_{lb} = \min_{Q \in A} G(p_l^Q) \quad (4.7)$$

The upper bound is:

$$c_{ub} = \max_{Q \in A} G(p_u^Q) \quad (4.8)$$

4.2 Poisson demand

In this section we will consider the situation where the demand has a discrete random variable that follows a Poisson distribution with parameter λ, $Poisson(\lambda)$. In the first case the parameter is known, as in the previous discussion, we can directly find the optimal order quantity and the expected total cost that the manager will infer.

But in the other case where the parameter λ (The mean number of successes that occur in a specified region) is not known, and we have a set of past demand samples, we need to use this set in order to estimate the parameter λ.

As discussed in the previous chapters, in order to estimate any parameter we can use the point estimation or the interval estimation.

Combine confidence interval analysis and inventory optimization

Since we have a set of past demand sample, we will use it to estimate the parameter λ by constructing a confidence interval.

Let $y_1,....y_m$ are the sample of a past demand for m days, using this data to compute a lower and upper bounds of the confidence interval for parameter λ.

We will construct an exact confidence interval for the unknown parameter with a confidence coefficient($1-\alpha$):

Since $y_i \sim Poisson(\lambda)$ so

$$x = \sum_{i=1}^{m} y_i \sim Poisson(m\lambda)$$

The bounds of the confidence interval for the parameter λ ,$(\lambda_{lb}, \lambda_{ub})$ are, respectively, the solution in λ to the following two equations:

$$\sum_{i=x}^{\infty} e^{-m\cdot\lambda} \frac{(m\cdot\lambda)^i}{i!} = \alpha / 2 \qquad (4.9)$$

$$\sum_{i=0}^{x} e^{-m\cdot\lambda} \frac{(m\cdot\lambda)^i}{i!} = \alpha / 2 \qquad (4.10)$$

Again, we can express in the terms of the chi-square distribution as discussed in section (3.2.2):

$$(\lambda_l, \lambda_u) = \left(\frac{1}{2m} \chi^2_{2x,1-\alpha/2}, \frac{1}{2m} \chi^2_{2(x+1),\alpha/2} \right) \qquad (4.11)$$

After we construct interval for the parameter λ , we need to determine a set of quantities that contains the optimal order quantity with confidence coefficient $1-\alpha$ and an interval for the estimated cost that the manager will infer whatever quantity he orders from the set of candidate quantities.

We will carry out this in a similar fashion to the binomial case that discussed in the previous section.

Let Q_{lb}^{*} be an optimal order quantity of the single period inventory under a Poisson demand $Poisson(\lambda_{lb})$.

And let Q_{ub}^{*} be an optimal order quantity of the single period inventory under a Poisson demand $Poisson(\lambda_{ub})$.

Now, we get, with confidence coefficient $1-\alpha$, a set that contains the optimal order quantity Q^{*} i.e.

$$Q^{*} \in A = \{Q_{lb}^{*}, Q_{lb}^{*}+1, \ldots\ldots, Q_{ub}^{*}-1, Q_{ub}^{*}\}.$$

Consider the cost associated with the order quantity Q under the Poisson demand $Poisson(\lambda)$.

$$E(G(Q)) = h\sum_{D=0}^{Q}(Q-D)e^{-\lambda}\frac{\lambda^{D}}{D!} + g\sum_{D=Q+1}^{\infty}(D-Q)e^{-\lambda}\frac{\lambda^{D}}{D!} \quad (4.12)$$

Consider the function:

$$G(\lambda) = h\sum_{D=0}^{Q}(Q-D)e^{-\lambda}\frac{\lambda^{D}}{D!} + g\sum_{D=Q+1}^{\infty}(D-Q)e^{-\lambda}\frac{\lambda^{D}}{D!} \quad (4.13)$$

In which the order quantity Q is fixed and the parameter λ is a variable.

Proposition: $G(\lambda)$ is a convex in the continuous parameter λ

PROOF:

Firstly, we can rewrite the function (4.13) as:

$$G(\lambda) = h(Q-\lambda) + (g+h)\sum_{i=Q}^{\infty}(1-F(i)) \qquad (4.14)$$

In order to prove $G(\lambda)$ is a convex we need to show $\frac{\partial^2 G(p)}{\partial p^2} \geq 0$ which is equivalent to $\frac{\partial^2}{\partial p^2}(\sum_{i=Q}^{n}(1-F(i))) \geq 0$ in other words we need to show $\frac{\partial^2}{\partial p^2}\sum_{i=Q}^{n}F(i) \leq 0$

We can rewrite $F(i) = F(i,\lambda) = \sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!}$

Proof:

$$\frac{\partial}{\partial \lambda}(\sum_{i=Q}^{\infty}\sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!}) = \sum_{i=Q}^{\infty}\sum_{k=0}^{i}(e^{-\lambda}\frac{k\lambda^{k-1}}{k!} + -e^{-\lambda}\frac{\lambda^k}{k!})$$

$$\frac{\partial^2}{\partial \lambda^2}(\sum_{i=Q}^{\infty}\sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!}) = \frac{\partial}{\partial \lambda}(\sum_{i=Q}^{\infty}\sum_{k=0}^{i}(e^{-\lambda}\frac{k\lambda^{k-1}}{k!} + -e^{-\lambda}\frac{\lambda^k}{k!}))$$

$$= \sum_{i=Q}^{\infty}\sum_{k=0}^{i}(e^{-\lambda}\frac{k(k-1)\lambda^{k-2}}{k!} - e^{-\lambda}\frac{k\lambda^{k-1}}{k!} - e^{-\lambda}\frac{k\lambda^{k-1}}{k!} + e^{-\lambda}\frac{\lambda^k}{k!})$$

$$= \sum_{i=Q}^{\infty}\sum_{k=0}^{i}(e^{-\lambda}\frac{k(k-1)\lambda^{k-2}}{k!} - 2e^{-\lambda}\frac{k\lambda^{k-1}}{k!} + e^{-\lambda}\frac{\lambda^k}{k!})$$

$$= \sum_{i=Q}^{\infty}(\sum_{k=0}^{i} e^{-\lambda}\frac{k(k-1)\lambda^{k-2}}{k!} - \sum_{k=0}^{i} 2e^{-\lambda}\frac{k\lambda^{k-1}}{k!} + \sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!})$$

$$= \sum_{i=Q}^{\infty}(\sum_{k=2}^{i} e^{-\lambda}\frac{\lambda^{k-2}}{(k-2)!} - \sum_{k=1}^{i} 2e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!})$$

$$= \sum_{i=Q}^{\infty}(\sum_{k=0}^{i-2} e^{-\lambda}\frac{\lambda^k}{k!} - \sum_{k=0}^{i-1} 2e^{-\lambda}\frac{\lambda^k}{k!} + \sum_{k=0}^{i} e^{-\lambda}\frac{\lambda^k}{k!})$$

$$= \sum_{i=Q}^{\infty}(F(i-2,\lambda) - 2F(i-1,\lambda) + F(i,\lambda))$$

$$= F(Q-2,\lambda) - 2F(Q-1,\lambda) + F(Q,\lambda) + F(Q-1,\lambda) - 2F(Q,\lambda) +$$
$$F(Q+1,\lambda) + F(Q,\lambda) - 2F(Q+1,\lambda) + F(Q+2,\lambda) + \ldots\ldots$$

$$= F(Q-2,\lambda) - F(Q-1,\lambda)$$

$$= \sum_{k=0}^{Q-2} e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=0}^{Q-1} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= -e^{-\lambda} \frac{\lambda^{Q-1}}{(Q-1)!} \leq 0$$

So the cost function is a convex in the continuous parameter λ. Therefore we will use the similar steps that discussed in the binomial demand to compute an interval that contains lower and upper bounds for the expected cost $(G(\lambda_l^Q), G(\lambda_u^Q))$ that the manager will face if he orders Q quantity by using unconstrained convex optimization approach. Also we can find the interval (c_{lb}, c_{ub}), with confidence coefficient $1-\alpha$, using the following formulas:

The lower bound is:

$$c_{lb} = \min_{Q \in A} G(\lambda_l^Q) \quad (4.15)$$

The upper bound is:

$$c_{ub} = \max_{Q \in A} G(\lambda_u^Q) \quad (4.16)$$

We will clarify steps in order to compute, under the confidence coefficient $1-\alpha$, a set of candidate order quantities and the cost that the manager will pay if he selects a quantity from this set.

1. From a set of past demand sample, employ confidence interval of the Poisson distribution as mentioned in equations (3.26) to find a lower and upper bounds for the parameter λ.

2. Determine the critical ratio R and then compute the bounds for the set A of order quantities.
3. For each element in A, compute an interval for the expected cost, and then compute a lower and upper bounds for the cost associated with these order quantities.

4.3 Exponential demand

Consider a continuous random demand that follows an Exponential distribution with parameter λ , $exponential(\lambda)$ .In the first case the parameter λ is known, as in the previous discussion, we can directly find the optimal order quantity by using equation (1.4) :

$$\int_0^{Q^*} f(D)dD = \frac{g}{h+g}$$

$$\int_0^{Q^*} \lambda e^{-\lambda D}\, dD = \frac{g}{h+g} \quad (4.17)$$

Such that h represents the unit overage cost and g represents the unit underage cost.

Then the optimal order quantity is given by: $Q^* = \frac{1}{\lambda}\ln(\frac{h+g}{h})$ (4.18)

We can compute the expected total cost that associated with any order quantity by using equation (2.2):

$$E(G(Q,D)) = \frac{h+g}{\lambda}(e^{-\lambda Q} + \frac{h}{h+g}(\lambda Q - 1)) \quad (4.19)$$

Also we can easily compute the expected total cost that associated with the optimal order quantity by substituting Q^* in the last equation and then we have:

$$E(G(Q^*,D)) = \frac{h}{\lambda} \cdot \ln(\frac{h+g}{h}) \quad (4.20)$$

Now we will consider another case where the parameter λ is not known, and we have a set of past demand samples, we will use this set in order to estimate the parameter λ.

Combine confidence interval analysis and inventory optimization

Since we have a set of past demand samples, we will use it to estimate the parameter λ by constructing a confidence interval.

Let $y_1,....y_m$ are the sample of a past demand for m- days, using this data to compute a lower and upper bounds of the confidence interval for the parameter λ in the exponential demand.

As discussed in section (3.2.3) we can construct an exact confidence interval for the unknown parameter with a confidence coefficient $1-\alpha$ by using this interval:

$$(\lambda_l, \lambda_u) = \left(\frac{\chi^2_{\alpha/2,2m}}{2\sum_{i=1}^{m} y_i} \ , \ \frac{\chi^2_{1-\alpha/2,2m}}{2\sum_{i=1}^{m} y_i} \right) \quad (4.21)$$

After constructing this interval, we will now determine an interval for quantities (Q_{lb}, Q_{ub}) that contains the optimal order quantity with confidence coefficient $1-\alpha$.

Let Q_{lb}^* is an optimal order quantity of the single period inventory under exponential demand $exponential(\lambda_{ub})$.

And let Q_{ub}^* is an optimal order quantity of the single period inventory under exponential demand $exponential(\lambda_{lb})$.

And we can find the values of Q_{lb}^* and Q_{ub}^* quantities as we clarify by using equation (4.18).

So after computing the lower and upper quantities we get, with confidence coefficient $1-\alpha$, an interval that includes the optimal order quantity Q^* i.e.

$$Q^* \in A = (Q_{lb}^*, Q_{ub}^*) .$$

At this point the manager can choice any quantity of this interval to order it at the beginning of the day. But he needs an information about the cost he will pay.

In other words we need to compute an interval for the expected estimated cost that the manager will face whatever the order quantity he chooses from the interval.

We will now construct a confidence interval, with confidence coefficient $1-\alpha$, of the expected total cost that the manager will pay whatever order quantity he orders from the interval A.

Recall the expected total cost associated with the order quantity Q is given by (4.19):

$$G(Q) = \frac{h+g}{\lambda}(e^{-\lambda Q} + \frac{h}{h+g}(\lambda Q - 1)) .$$

Unfortunately, $G(Q)$ is not convex in the continuous parameter λ so as discussed in [35], we can compute an interval for the expected cost associated with any quantity that the manager chooses from A by using:

The lower bound is:

$$c_{lb}^{*} = G(Q_{lb}^{*}, \lambda_{ub}) \qquad (4.22)$$

And the upper bound is:

$$c_{ub}^{*} = \max\{G(Q_{lb}^{*}, \lambda_{lb}), G(Q_{ub}^{*}, \lambda_{ub})\} \qquad (4.23)$$

Chapter Five

Comparison between confidence approach and point estimation approach

In this chapter we will present algorithms that facilitate dealing with each demand distribution that discussed in the previous chapters in order to identify a range of order quantities that, with confidence coefficient $1-\alpha$, includes the real optimal order quantity, and in order to produce an interval for the expected cost associated with the range of order quantities.

5.1Binomial demand

Consider the issue in single period single item inventory control problem, let h be the unit overage cost, paid for each item left in stock after demand realized, and let g be the unit underage cost, paid for each unit not achieved demand, and let the demand has a Binomial distribution with parameters n and p, in which the parameter (probability of success) p is not known. In the first algorithm we will employ confidence interval, with confidence coefficient $1-\alpha$, in order to find a range of order quantities and interval for the associated cost, and in the second algorithm using point estimation instead of interval estimation.

Recall:

The confidence interval for the parameter p is given by using quantiles from the beta distribution:

$$(p_{lb}, p_{ub}) = (beta(\alpha/2, x, n\cdot m - x + 1), beta(1-\alpha/2, x+1, n\cdot m - x))$$

The expected total cost associated with the order quantity Q under the binomial random demand, $binom(n, p)$ is given by :

$$G(p) = h\sum_{D=0}^{Q}(Q-D)\binom{n}{D}p^D(1-p)^{n-D} + g\sum_{D=Q+1}^{n}(D-Q)\binom{n}{D}p^D(1-p)^{n-D}$$

The optimal order quantity of the single period inventory under binomial demand $binom(n, p)$ with probability success p is given by:

$Q = Inverse\,CDF(binom(n, p), R)$, where R is the critical fractile.

Algorithm 1.1: Single period inventory model with Binomial demand (confidence approach).

Input: confidence coefficient $1-\alpha$

the unit overage: h

the unit underage: g

the number of customers per time: n

the number of past demand sample: m

a set of past demand sample $\{d_1, ..., d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find the solutions of the equations 4.1 and 4.2 to get the bounds of the confidence interval (p_{lb}, p_{ub})

Step 3 determine the critical ratio $R = \dfrac{g}{g+h}$

Step 4 determine $Q_{lb} : (Q_{lb} = Inverse\,CDF(binom(n, p_{lb}), R))$

determine $Q_{ub} : (Q_{ub} = Inverse\,CDF(binom(n, p_{ub}), R))$

$A = \{Q_{lb}, ..., Q_{ub}\}$

Step 5 for each $Q \in A$ repeat step 5.1-5.4 until end of the set

Step 5.1 find p_u^Q that maximize $G(p)$ in the interval (p_{lb}, p_{ub})

Step 5.2 find p_l^Q that minimize $G(p)$ in the interval (p_{lb}, p_{ub})

Step 5.3 find $G(p_l^Q)$

Step 5.4 find $G(p_u^Q)$

Step 6 find the lower bound of the interval of the estimated expected total cost

$$c_{lb} = \min_{Q \in A} G(p_l^Q)$$

Step 7 find the upper bound of the interval of the estimated expected total cost

$$c_{ub} = \max_{Q \in A} G(p_u^Q)$$

Output: the set A of candidate order quantities.

the interval of the estimated expected total cost.

Algorithm 1.2: Single period inventory model with Binomial demand (point estimation approach).

Input: the unit overage: h

the unit underage: g

the number of customers per time: n

the number of past demand sample: m

a set of past demand sample $\{d_1, ..., d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find an approximation for the probability of success using any methods of the point estimation, e.g. using MLE we can find $\hat{p}$ such that

$$\hat{p} = \frac{1}{m \cdot n} \cdot \sum_{i=1}^{m} d_i \quad .$$

Step 3 determine the critical ratio $R = \frac{g}{g+h}$

Step 4 determine $\hat{Q}$: $(\hat{Q} = Inverse\,CDF(binom(n, \hat{p}), R))$

Step 5 find the estimated expected total cost for $\hat{Q}$: $G(\hat{p})$

Step 6 find $p_u^{\hat{Q}}$ that maximize $G(p)$.

Step 7 find $p_l^{\hat{Q}}$ that minimize $G(p)$.

Step 8 find the interval of the estimated expected total cost for $\hat{Q}$:

Step 8.1 find $G(p_l^{\hat{Q}})$

Step 8.2 find $G(p_u^{\hat{Q}})$

Output: the candidate optimal order quantity $\hat{Q}$.

the estimated expected total cost for $\hat{Q}$.

the interval of the estimated expected total cost for $\hat{Q}$.

5.2 Poisson demand:

Consider the issue in a single period single item inventory control problem, let h be the unit overage cost, paid for each item left in stock after demand realized, and let g be the unit underage cost, paid for each unit not achieved demand, and let the demand has a Poisson distribution with unknown parameter λ. In the first algorithm we will employ confidence interval, with confidence coefficient $1-\alpha$, in order to find a range of order quantities and interval for the associated cost, and in the second algorithm using point estimation instead of interval estimation.

Recall:

The confidence interval for the parameter λ can give by using quantiles from the chi-square distribution:

$$(\lambda_l, \lambda_u) = \left(\frac{1}{2m} \chi^2_{2x,1-\alpha/2}, \frac{1}{2m} \chi^2_{2(x+1),\alpha/2} \right)$$

The expected total cost associated with the order quantity Q under the binomial random demand, $Poisson(\lambda)$ is given by:

$$G(\lambda) = h \sum_{D=0}^{Q} (Q-D) e^{-\lambda} \frac{\lambda^D}{D!} + g \sum_{D=Q+1}^{\infty} (D-Q) e^{-\lambda} \frac{\lambda^D}{D!}$$

The optimal order quantity of the single period inventory under Poisson demand $Poisson(\lambda)$ is given by:

$Q = Inverse\,CDF(Poisson(\lambda), R)$, where R is the critical fractile.

Algorithm 2.1: Single period inventory model with Poisson demand (confidence approach).

Input: confidence coefficient $1-\alpha$

the unit overage: h

the unit underage: g

the number of past demand sample: m

a set of past demand sample $\{d_1, \ldots\ldots, d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find the solutions of the equations 4.9 and 4.10 to get the bounds of

the confidence interval $(\lambda_{lb}, \lambda_{ub})$

Step 3 determine the critical ratio $R = \frac{g}{g+h}$

Step 4 determine $Q_{lb} : (Q_{lb} = Inverse\,CDF(Poisson(\lambda_{lb}), R))$

determine $Q_{ub} : (Q_{ub} = Inverse\,CDF(Poisson(\lambda_{ub}), R))$

$A = \{Q_{lb}, ..., Q_{ub}\}$

Step 5 for each $Q \in A$ repeat step 5.1-5.4 until end of the set

Step 5.1 find λ_u^Q that maximize $G(\lambda)$ in the interval $(\lambda_{lb}, \lambda_{ub})$

Step 5.2 find λ_l^Q that minimize $G(\lambda)$ in the interval $(\lambda_{lb}, \lambda_{ub})$

Step 5.3 find $G(\lambda_l^Q)$

Step 5.4 find $G(\lambda_u^Q)$

Step 6 find the lower bound of the interval of the estimated expected total cost

$$c_{lb} = \min_{Q \in A} G(\lambda_l^Q)$$

Step 7 find the upper bound of the interval of the estimated expected total cost

$$c_{ub} = \max_{Q \in A} G(\lambda_u^Q)$$

Output: the set A of candidate order quantities.

the interval of the estimated expected total cost.

Algorithm2.2: Single period inventory model with Poisson demand (Point estimation approach).

Input: the unit overage: h

the unit underage: g

the number of past demand sample: m

a set of past demand sample $\{d_1, ..., d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find an approximation for the parameter λ using any methods of the point estimation, e.g. using MLE we can find $\hat{\lambda}$ such that $\hat{\lambda} = \frac{1}{m} \cdot \sum_{i=1}^{m} d_i$.

Step 3 determine the critical ratio $R = \frac{g}{g+h}$

Step 4 determine $\hat{Q}$: ($\hat{Q} = Inverse\,CDF(Poisson(\hat{\lambda}), R)$)

Step 5 find the estimated expected total cost for $\hat{Q}$: $G(\hat{\lambda})$

Step 6 find $\lambda_u^{\hat{Q}}$ that maximize $G(\lambda)$.

Step 7 find $\lambda_l^{\hat{Q}}$ that minimize $G(\lambda)$.

Step 8 find the interval of the estimated expected total cost for $\hat{Q}$:

Step 8.1 find $G(\lambda_l^{\hat{Q}})$

Step 8.2 find $G(\lambda_u^{\hat{Q}})$

Output: the candidate optimal order quantity $\hat{Q}$.

the estimated expected total cost for $\hat{Q}$.

the interval of the estimated expected total cost for $\hat{Q}$.

5.3 Exponential demand:

Consider the issue in single period single item inventory control problem, let h be the unit overage cost, paid for each item left in stock after demand realized, and let g be the unit underage cost, paid for each unit not achieved demand, and let the demand has an Exponential distribution with unknown parameter λ. In the first algorithm we will employ confidence interval, with confidence coefficient $1-\alpha$, in order to find a range of order quantities and confidence interval for the associated cost, and in the second algorithm using point estimation instead of interval estimation.

Recall

$$Q^* = \frac{1}{\lambda}\ln(\frac{h+g}{h})$$

The expected total cost associated with the order quantity Q is given by:

$$G(Q,\lambda) = \frac{h+g}{\lambda}(e^{-\lambda Q} + \frac{h}{h+g}(\lambda Q - 1))$$

Algorithm 3.1: Single period inventory model with Exponential demand (confidence approach).

Input: confidence coefficient $1-\alpha$

the unit overage: h

the unit underage: g

the number of past demand sample: m

a set of past demand sample $\{d_1, \ldots\ldots, d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find the bounds of the unknown parameter $(\lambda_{lb}, \lambda_{ub})$ by using equation (4.21)

Step 3 determine the critical ratio $R = \frac{g}{g+h}$

Step 4 determine Q_{lb} : ($Q_{lb} = \frac{1}{\lambda_{ub}}\ln(\frac{h+g}{h})$)

determine Q_{ub} : ($Q_{ub} = \frac{1}{\lambda_{lb}} \ln(\frac{h+g}{h})$)

$A = (Q_{lb}, Q_{ub})$

Step 5 find the lower bound of the estimated expected total cost:

$$c^*_{lb} = G(Q^*_{lb}, \lambda_{ub})$$

Step 6 find the upper bound of the estimated expected total cost:

$$c^*_{ub} = \max\{G(Q^*_{lb}, \lambda_{lb}), G(Q^*_{ub}, \lambda_{ub})\}$$

Output: the interval A of candidate order quantities.

the interval of the estimated expected total cost.

Algorithm3.2: Single period inventory model with Exponential demand

(Point estimation approach).

Input: the unit overage: h

the unit underage: g

the number of past demand sample: m

a set of past demand sample $\{d_1, ..., d_m\}$

Step 1 calculate the summation of past demand sample: $\sum_{i=1}^{m} d_i$

Step 2 find an approximation for the parameter λ using any methods

of the point estimation, e.g. using MLE we can find $\hat{\lambda}$ such that

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}.$$

Step 3 determine the critical ratio $R = \frac{g}{g+h}$

Step 4 determine $\hat{Q}$: ($\hat{Q} = \frac{1}{\hat{\lambda}} \ln(\frac{h+g}{h})$)

Step 5 find the estimated expected total cost for $\hat{Q} : c^* = G(\hat{Q}, \hat{\lambda})$

Output: the candidate optimal order quantity $\hat{Q}$.

the estimated expected total cost for $\hat{Q}$.

5.4 Examples and discussion

Example: Binomial demand

Consider a newsvendor problem under a Binomial demand with $n = 50$ and with unknown parameter p, assume $h = 1$ and $g = 3$ also the manager is given 10 samples of past demand. The samples are: $\{28, 27, 25, 23, 29, 26, 28, 28, 22, 28\}$, we will use this samples and after using the algorithms 1.1 and 1.2 we will determine the candidate optimal quantity and the associated estimated expected cost (Let confidence coefficient $1-\alpha = 0.9$).

Binomial demand	Point estimation approach						Confidence interval approach		
	MLE			Bayesian					
	$\hat{p}$	$\hat{Q}$	$G(\hat{p})$	$\hat{p}$	$\hat{Q}$	$G(\hat{p})$	(p_{lb}, p_{ub})	$\{Q_{lb},....,Q_{ub}\}$	(c_{lb}, c_{ub})
	0.528	29	4.46149	0.528	29	4.46149	(0.489,0.5657)	{27,…,31}	(4.4269,7.2295)

Example: Poisson demand

Consider a newsvendor problem under a Poisson demand with unknown parameter λ, assume $h = 1$ and $g = 3$ also the manager is given 10 samples of past demand. The samples are: $\{51, 55, 49, 45, 52, 41, 51, 54, 50, 39\}$ we will use this samples and after using the algorithms 2.1 and 2.2 we will determine the candidate optimal quantity and the associated estimated expected cost (Let confidence coefficient $1-\alpha = 0.9$).

Poisson demand	Point estimation approach						Confidence interval approach		
	MLE			Bayesian					
	$\hat{\lambda}$	$\hat{Q}$	$G(\hat{\lambda})$	$\hat{\lambda}$	$\hat{Q}$	$G(\hat{\lambda})$	$(\lambda_{lb},\lambda_{ub})$	$\{Q_{lb},....,Q_{ub}\}$	(c_{lb},c_{ub})
	48.7	53	9.23329	48.7	53	9.23329	(45.13,52.49)	{50,…,57}	(8.86,14.62)

From these results we can build visualize about the decisions that the manager will choose in order to achieve to the optimal profit.

In the confidence interval approach, we make an interval for the unknown parameter and this interval will cover the actual value according to the prescribed confidence probability. The size of the confidence interval is controlled by the manager through changing the sample size or changing the confidence level.

We can note that the confidence interval is being independent of a prior information about the unknown parameter whereas the Bayesian method is depend on a prior information.

Depending on the interval of the unknown parameter we can immediately build a confidence interval for the order quantity and the associated cost.

If the manager in a risk-taker, he has a better control of exceeding a certain cost and a perfect expectation about the range of order quantities.

On the other hand, if the manager is not a risk-taker, he can select the order quantity $\hat{Q}$ (from the point estimation approach) and find the interval of the expected estimated cost as discussed in the confidence interval approach (i.e. in the binomial demand we will find the interval $(G(p_l^{\hat{Q}}), G(p_u^{\hat{Q}}))$.

Example: Exponential demand

Consider the situation in the second example in section 3.1.1 except that demand has an Exponential distribution for which the parameter λ is unknown, in which $h=1$and $g=3$ the manager is given 10 samples from the past demand, and the samples are:

$\{39.79, 39.26, 32.21, 0.51, 107.03, 72.87, 45.23, 20.12, 26.46, 56.8\}$

(Let confidence coefficient$1-\alpha=0.9$).

By using Algorithm 3.1 we compute the $1-\alpha$ confidence interval for the parameterλ, and confidence interval for the order quantities that -with $1-\alpha$ confidence coefficient – contains the optimal order quantity, and also the confidence interval for the estimated expected total cost.

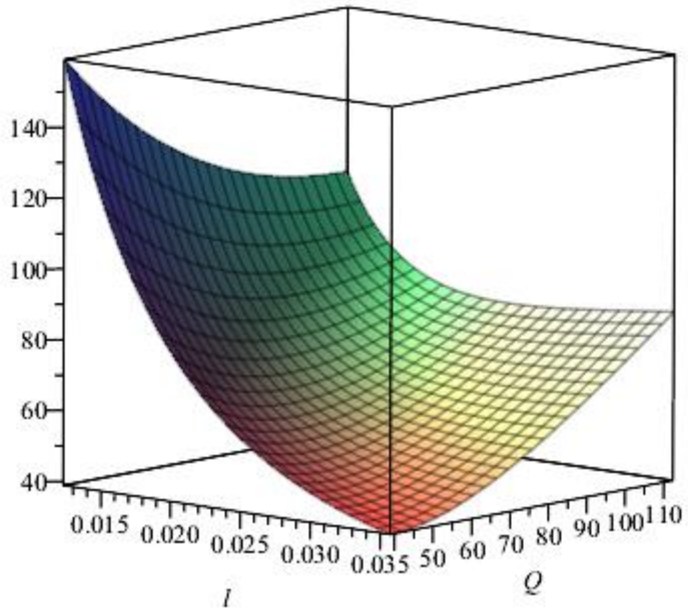
By using Algorithm 3.2 we compute the estimator of the parameterλ, and depending on this value we compute the order quantity that, and also the associated expected total cost. Table 3 summarize the results from applying the two Algorithms.

Exponential demand	Point estimation approach					
	MLE			Bayesian		
	$\hat{\lambda}$	$\hat{Q}$	$G(\hat{Q},\hat{\lambda})$	$\hat{\lambda}$	$\hat{Q}$	$G(\hat{Q},\hat{\lambda})$
	0.0227	61.0358	61.0358	0.0283	47.0406	49.0406

Exponential demand	Confidence interval approach		
	$(\lambda_{lb},\lambda_{ub})$	(Q_{lb},Q_{ub})	(c_{lb},c_{ub})
	(0.0123,0.0357)	(38.87,112.49)	(38.87,112.514)

In the following Figure we provide outlook of the expected cost as a function of λ and Q.

Figer: Expected total cost as a function of λ and of Q



From this figure we note that

$$c_{lb}^{*} = G(Q_{lb}^{*}, \lambda_{ub})$$

And that:

$$c_{ub}^{*} = \max\{G(Q_{lb}^{*}, \lambda_{lb}), G(Q_{ub}^{*}, \lambda_{ub})\} = G(Q_{lb}^{*}, \lambda_{lb})$$

As we discussed in the discrete case, in the confidence interval approach, we make an interval for the unknown parameter and this interval will cover the actual value according to the prescribed confidence probability. We can note that the confidence interval is being independent of a prior information about the unknown parameter whereas the Bayesian method is depend on a prior information.

Depending on the interval of the unknown parameter we can immediately build a confidence interval for the order quantity and the associated cost.

If the manager in a risk-taker, he has a better control of exceeding a certain cost and a perfect expectation about the confidence of order quantities.

On the other hand, if the manager is not a risk-taker, he can select the order quantity $\hat{Q}$ (from the point estimation approach) and find the interval of the expected estimated cost.

Our analysis is limited to three distributions: in the Binomial distribution we know all of the information about the random demand that the Binomial distribution has a positive mean and takes a discrete values from zero to n, the same things about the Poisson distribution but takes a discrete values from zero to infinity.

Also in the Exponential distribution we know all of the information about the random demand that the Exponential distribution has a positive mean and takes a continuous values from zero to infinity.

And we limit our work on analysis the case in which a single parameter that must be estimated, and we use the exact confidence intervals and leave the analysis of using an approximate intervals.

Conclusion

In this thesis, we have studied the problem of controlling the inventory of a single item over a single period with stochastic demand in which the distribution of the demand has an unknown parameter.

We introduced a method of estimating the unknown parameter using the confidence interval and depending on the results from estimating the unknown parameter we identify a range of order quantities that-with $1-\alpha$ confidence coefficient – contains the optimal order quantity, and then we build an interval for the estimated expected cost that the manager will pay if he orders any quantity from the range that we constructed in the previous step.

Also, we introduced a method of estimating the unknown parameter using the point estimation, and depending on the results from estimating the unknown parameter we identify a candidate optimal order quantity under the estimated parameter, and then we find an estimated expected cost that the manager will pay if he orders this quantity. However this method does not provide any information on the reliability of the estimation.

We considered three cases, the demand has a Binomial distribution with unknown parameter p , and the demand has a Poisson distribution with unknown parameter λ, also we consider the case in which the demand has an Exponential distribution with unknown parameter λ .For each of these cases we use the exact confidence interval approach and the point estimation approach.

We presented numerical examples in order to clarify our strategy and to show how the confidence interval approach complements with the point estimation approach in order to give the best outlook to the manager to take a decision that achieve an optimal profit -with $1-\alpha$ confidence coefficient- so, If the manager in a risk-taker, he has a better control of exceeding a certain cost and a perfect expectation about the range of order quantities.

On the other hand, if the manager is not a risk-taker, he can select the order quantity $\hat{Q}$ (from the point estimation approach) and find the interval of the expected estimated cost as discussed in the confidence interval approach.

The approach we considered does not simply provide point estimation; it provides instead complete information to the decision maker about the set of potentially optimal order quantities according to the available data and to the chosen confidence level and about the confidence interval for the expected cost associated with each of these quantities.

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جامعة النجاح الوطنية
كلية الدراسات العليا

إيجاد فترة ثقة تحتوي على الحل الأمثل لنموذج تخزين لفترة واحدة

إعداد
ثناء حسام الدين أمين أبوصاع

اشراف
د.محمد نجيب اسعد

قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات المحوسبة بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس-فلسطين
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الملخص

هدفت هذه الدراسة الى معالجة مشكلة تقدير الطلب لتحديد كمية المخزون المثلى لتحقيق اقل تكلفة و بأقصى ربح لنموذج تخزين نوع وحيد لفترة وحيدة مع فرضية ان توزيع الطلب معروف لكن أحد معالمه غير معروفة.

أثناء معالجة هذه المشكلة فرضنا ان صاحب قرار تحديد كمية المخزون يملك عينة من الطلب من الايام الماضية وأيضا يعلم نوع توزيع الطلب لكن احد معالم التوزيع غير معروفه.

قدمنا طريقتين لتقدير المعلمة غير المعروفة ؛ الطريقة الأولى كانت تعتمد على ايجاد نقطة تقديرية للمعلمة المجهولة. فيما كانت الطريقة الثانية تعتمد على ايجاد فترات ثقة تحتوي المعلمة المجهولة بمعامل ثقة $(1-\alpha)$

بناءا على طريقة تقدير المعلمة ، قمنا بإيجاد مجموعة من الكميات تحتوي على الكمية المثلى التي تحقق اقل تكلفة بمعامل ثقة $(1-\alpha)$ ثم اوجدنا فترة ثقة للتكلفة المتوقع ان يدفعها صاحب القرار اذا طلب احدى الكميات السابقة بمعامل ثقة $(1-\alpha)$.

طبّقنا هاتين الطريقتين على ثلاث انواع من التوزيعات ثم قدمنا امثلة عددية لتوضيح التكامل بين طريقتي تقدير المعلمة وايجاد فترات الثقة لإيجاد الكمية المثلى لتحقيق اقل تكلفة.